11 – 19 PROGRESSION

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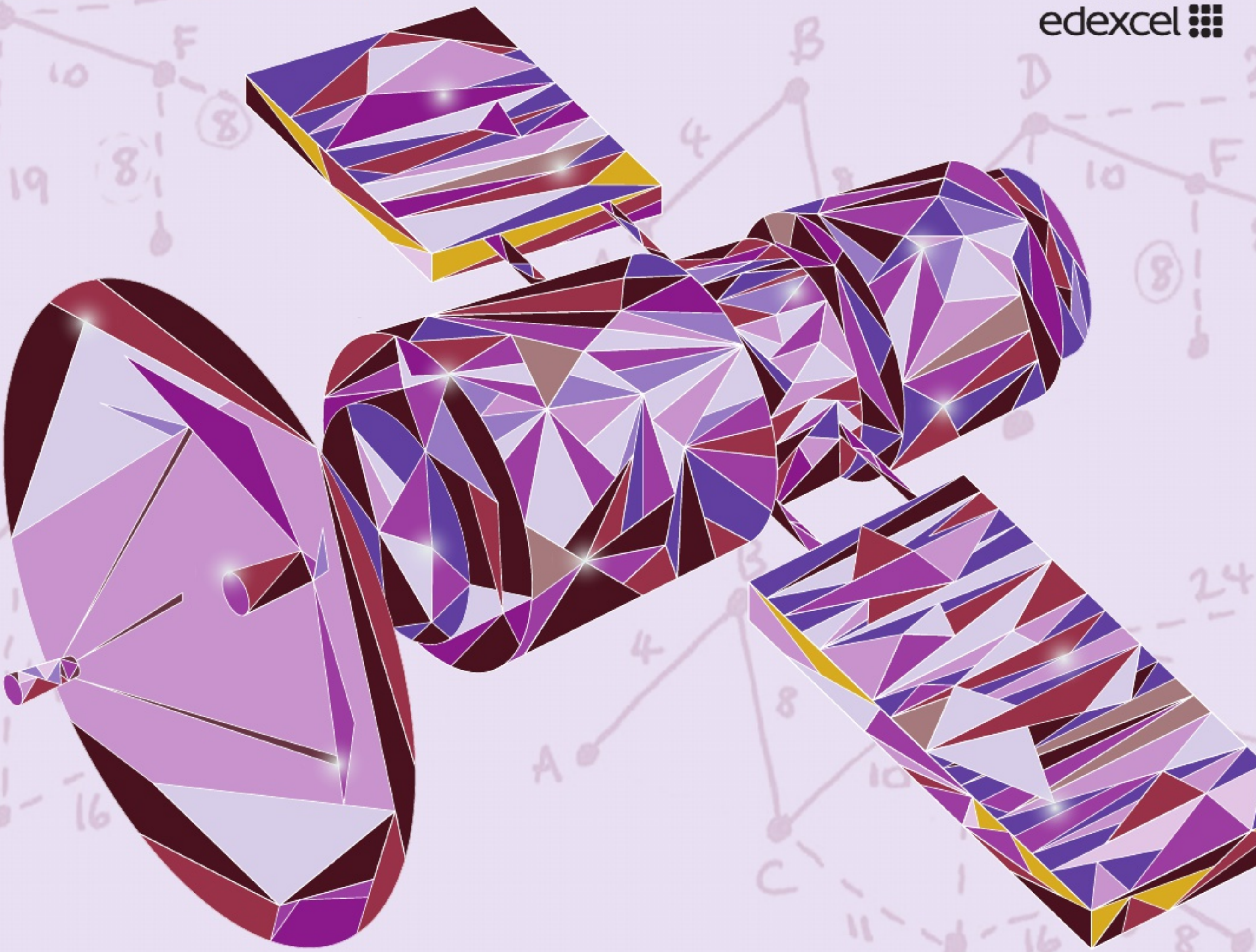

Edexcel AS and A level Further Mathematics

# Decision Mathematics 2

# D2

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Pearson

Published by Pearson Education Limited, 80 Strand, London WC2R 0RL.

www.pearsonschoolsandfecolleges.co.uk

Copies of official specifications for all Pearson qualifications may be found on the website: qualifications.pearson.com


Edited by Tech-Set Ltd, Gateshead
Typeset by Tech-Set Ltd, Gateshead

Cover illustration Marcus@kja-artists



First published 2018

21 20 19 18
10 9 8 7 6 5 4 3 2 1

**British Library Cataloguing in Publication Data**
A catalogue record for this book is available from the British Library

ISBN 978 1292 18330 5



Printed in the UK by Bell and Bain Ltd, Glasgow

**Acknowledgements**
The authors and publisher would like to thank the following for their kind permission to reproduce their photographs:

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# Contents

● = A level only

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- Challenge boxes provide extra stretch

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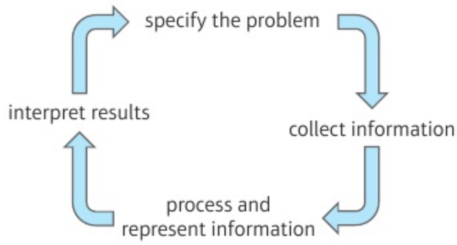


**3. Mathematical modelling**

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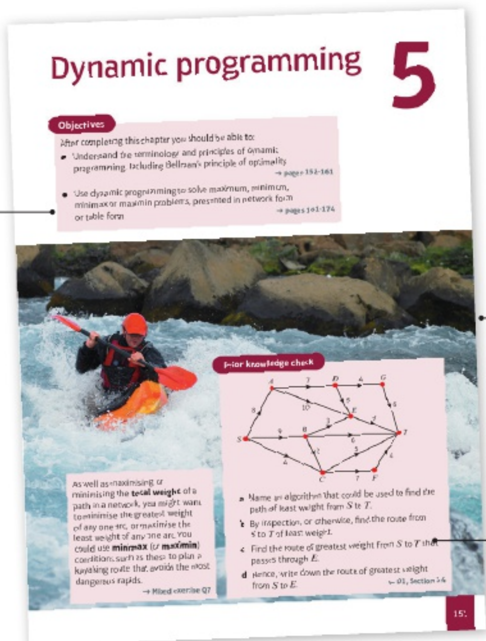


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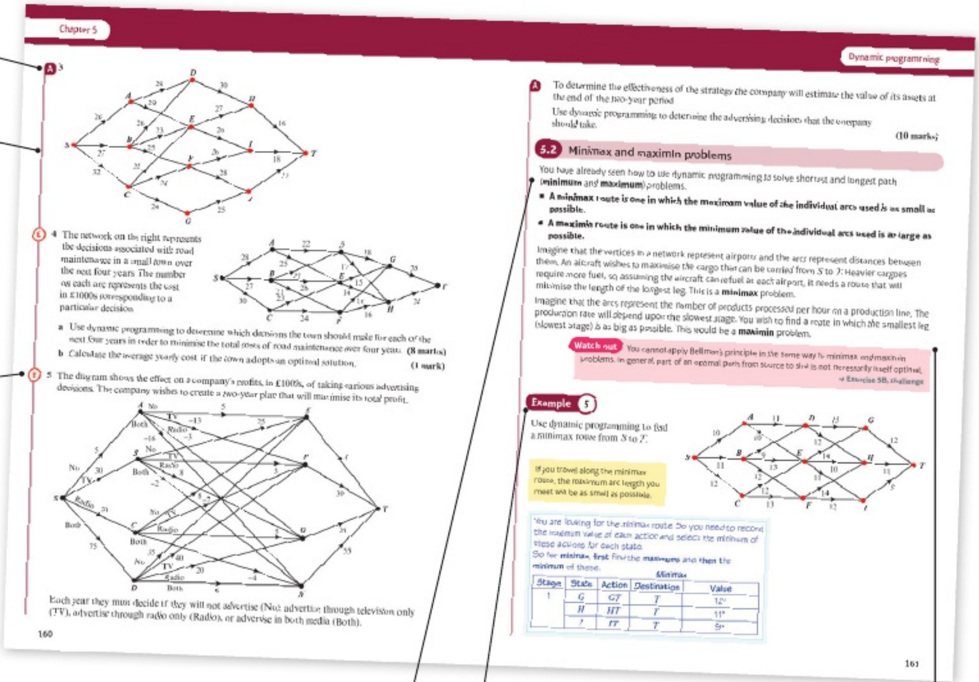
The *Prior knowledge check* helps make sure you are ready to start the chapter

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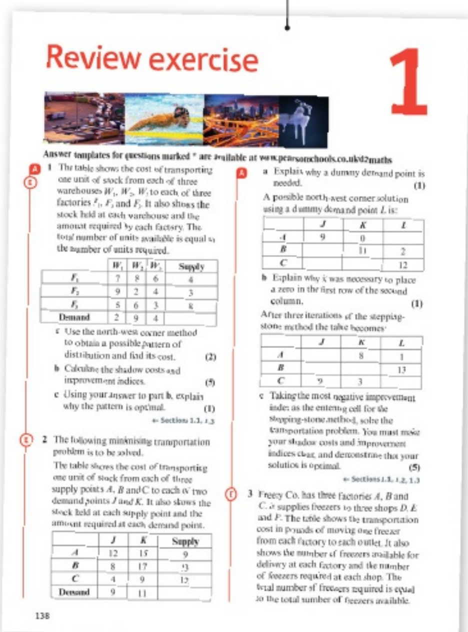


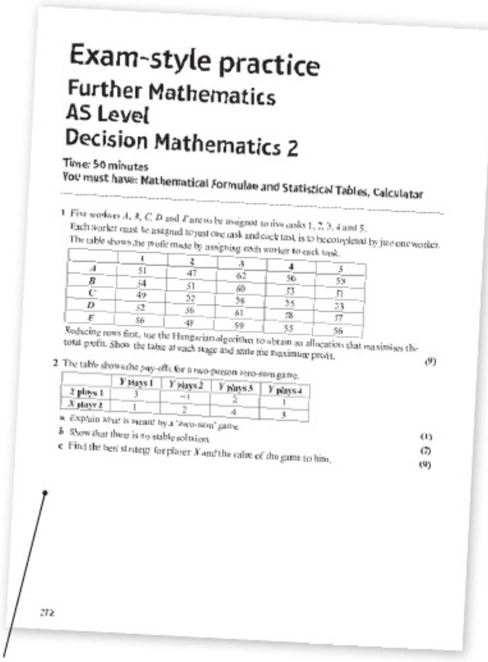


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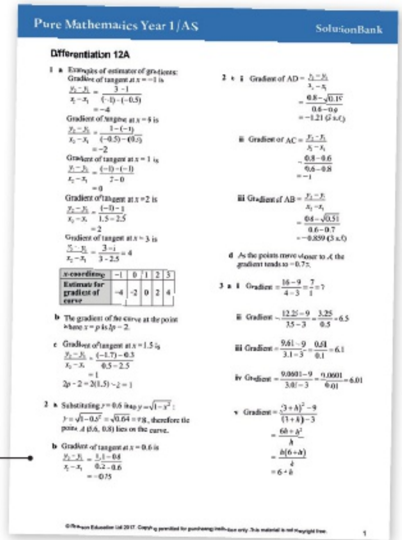
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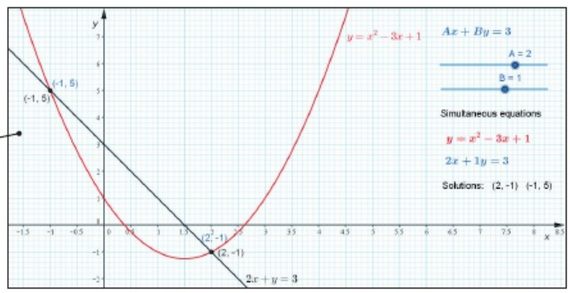


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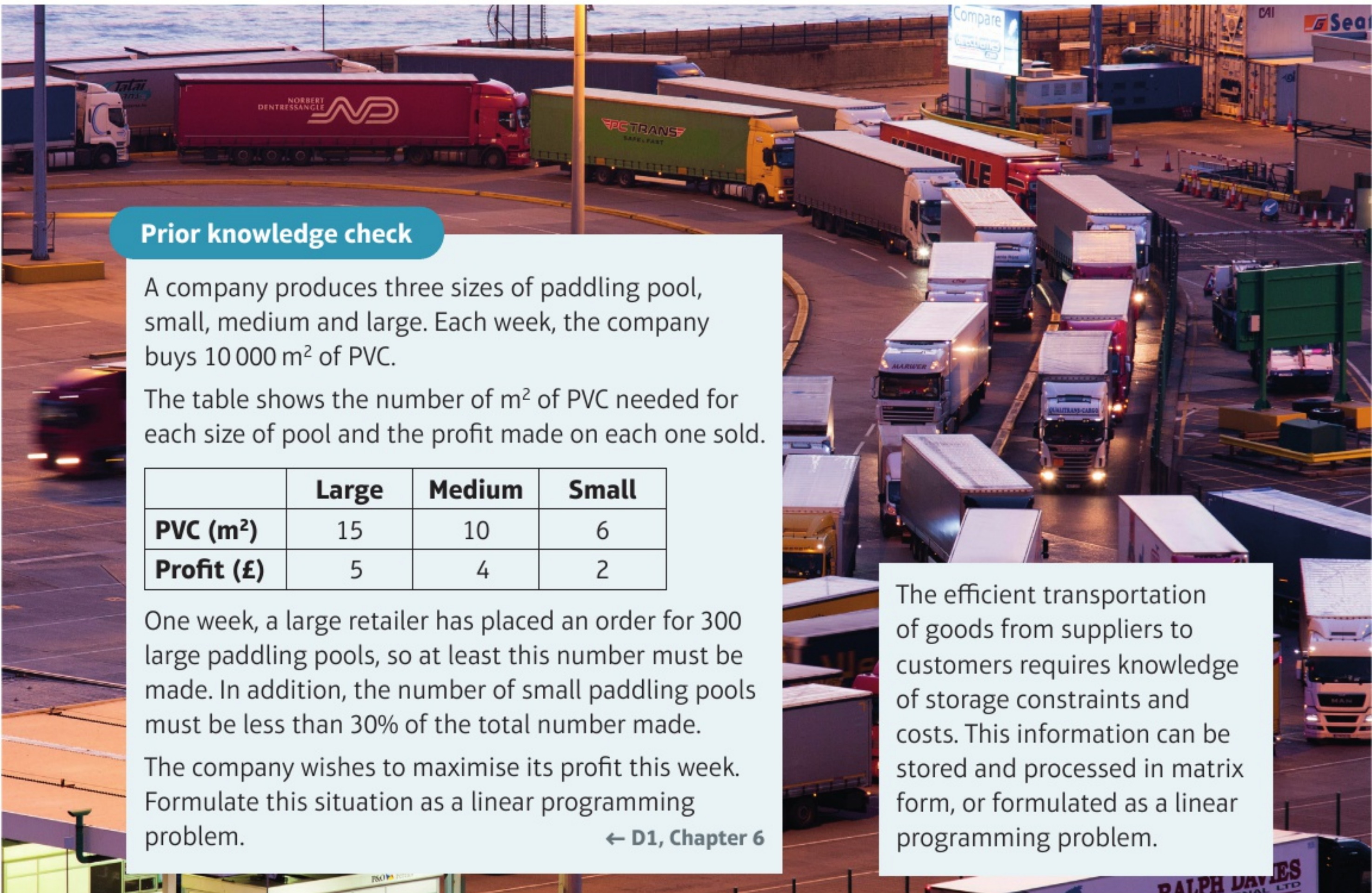
# Transportation problems

# 1

## Objectives

After completing this chapter you should be able to:

- Describe and model transportation problems → pages 2–37
- Use the north-west corner method → pages 2–6
- Understand unbalanced transportation problems and degenerate solutions → pages 6–11
- Use shadow costs to find improvement indices → pages 11–18
- Use the stepping-stone method → pages 19–28
- Formulate a transportation problem as a linear programming problem → pages 28–35




## Prior knowledge check

A company produces three sizes of paddling pool, small, medium and large. Each week, the company buys 10 000 $m^2$ of PVC.

The table shows the number of $m^2$ of PVC needed for each size of pool and the profit made on each one sold.

| | Large | Medium | Small |
|---|---|---|---|
| **PVC ($m^2$)** | 15 | 10 | 6 |
| **Profit (£)** | 5 | 4 | 2 |

One week, a large retailer has placed an order for 300 large paddling pools, so at least this number must be made. In addition, the number of small paddling pools must be less than 30% of the total number made.

The company wishes to maximise its profit this week. Formulate this situation as a linear programming problem. ← D1, Chapter 6

The efficient transportation of goods from suppliers to customers requires knowledge of storage constraints and costs. This information can be stored and processed in matrix form, or formulated as a linear programming problem.

## 1.1 The north-west corner method

In this chapter you will look at the costs relating to the transportation of goods: to factories, and from factories to warehouses and customers. The problems considered are usually concerned with minimising distribution costs in situations where there are multiple sources and multiple destinations. You need to be familiar with the terminology used in describing and modelling transportation problems:

- **The capacity of each of the supply points (or sources) – the quantity of goods that can be produced at each factory or held at each warehouse. This is called the supply or stock.**
- **The amount required at each of the demand points (or destinations) – the quantity of goods that are needed at each shop or by each customer. This is called the demand.**
- **The unit cost of transporting goods from the supply points to the demand points.**

The unit cost is the cost of transporting one item. If the unit cost is £$c$ then it will cost £$nc$ to transport $n$ items.

### Example 1

Three suppliers $A$, $B$ and $C$ each produce road grit which has to be delivered to council depots $W$, $X$, $Y$ and $Z$. The stock held at each supplier and the demand from each depot is known. The cost, in pounds, of transporting one lorry load of grit from each supplier to each depot is also known. This information is given in the table.

| | Depot $W$ | Depot $X$ | Depot $Y$ | Depot $Z$ | Stock (lorry loads) |
|---|---|---|---|---|---|
| **Supplier $A$** | 180 | 110 | 130 | 290 | 14 |
| **Supplier $B$** | 190 | 250 | 150 | 280 | 16 |
| **Supplier $C$** | 240 | 270 | 190 | 120 | 20 |
| **Demand (lorry loads)** | 11 | 15 | 14 | 10 | 50 |

This table is often referred to as the cost matrix.

This is the cost in pounds of transporting one lorry load from $B$ to $Y$. This cost is sometimes written as $C_{BY}$ or $C(BY)$.

Notice that the total supply is equal to the total demand. If this is not the case we introduce a dummy destination to absorb the excess supply, with transportation costs all zero. → Example 3

Use the information in the table to write down:

**a** the number of lorry loads of grit that each supplier can supply
**b** the number of lorry loads of grit required at each depot
**c** the cost of transporting a lorry load of grit from $A$ to $W$
**d** the cost of transporting a lorry load of grit from $C$ to $Z$.
**e** Which is the cheapest route to use?
**f** Which is the most expensive route to use?

**a** Suppliers $A$, $B$ and $C$ can provide 14, 16 and 20 lorry loads respectively.
**b** Depots $W$, $X$, $Y$ and $Z$ require 11, 15, 14 and 10 lorry loads respectively.
**c** The cost of transporting one lorry load from $A$ to $W$ is £180.
**d** The cost of transporting one lorry load from $C$ to $Z$ is £120.
**e** The cheapest route is $A$ to $X$ at £110 per load.
**f** The most expensive route is $A$ to $Z$ at £290 per load.

A In general, the manufacturer would like to minimise the total transportation costs whilst still meeting demand. This is called the **transportation problem** and can be solved using the **transportation algorithm**:

**The transportation algorithm**

1. First find an initial solution that uses all the stock and meets all the demands.
2. Calculate the total cost of this solution and see if it can be reduced by transporting some goods along a route not currently in the solution. (If this is not possible then the solution is optimal.)
3. If the cost can be reduced by using a new route, allocate as many units as possible to this new route to create a new solution.
4. Check the new solution in the same way as the initial solution to see if it is optimal. If not, repeat step 3.
5. When no further savings are possible, an optimal solution has been found.

The first step in the box above requires you to find an **initial solution**. The method used for this is called the **north-west corner method**:

**Notation** The solution generated by this method is sometimes called the initial (or basic) **feasible** solution.

**The north-west corner method**

1. Create a table, with one row for every source and one column for every destination. Each cell represents a route from a source to a destination. Each destination's demand is given at the foot of each column and each source's stock is given at the end of each row. Enter numbers in each cell to show how many units are to be sent along that route.
2. Begin with the top left-hand corner (the north-west corner). Allocate the maximum available quantity to meet the demand at this destination (whilst not exceeding the stock at this source).
3. As each stock is emptied, move one square down and allocate as many units as possible from the next source until the demand of the destination is met. As each demand is met, move one square to the right and again allocate as many units as possible.
4. When all the stock is assigned, and all the demands are met, stop.

**Watch out** In order to avoid degenerate solutions, movements are made between squares either vertically or horizontally but **never** diagonally. → **Example 5**

**Note** For a problem involving $m$ sources and $n$ destinations, you must enter $n + m - 1$ transportation quantities $\geqslant 0$. This will not reduce throughout the problem.

**Note** In your exam, problems will be restricted to a maximum of 4 sources and 4 destinations.

## Example 2

| | Depot $W$ | Depot $X$ | Depot $Y$ | Depot $Z$ | Stock |
|---|---|---|---|---|---|
| Supplier $A$ | 180 | 110 | 130 | 290 | 14 |
| Supplier $B$ | 190 | 250 | 150 | 280 | 16 |
| Supplier $C$ | 240 | 270 | 190 | 120 | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

Use the north-west corner method to find an initial solution to the problem described in Example 1 and shown in the table, and state its cost.

**Online** Explore how the north-west corner method can be used to find an initial solution using GeoGebra.

Units transported

| | $W$ | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|---|
| $A$ | | | | | 14 |
| $B$ | | | | | 16 |
| $C$ | | | | | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

**1** Set up the table.

| | $W$ | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|---|
| $A$ | 11 | | | | 14 |
| $B$ | | | | | 16 |
| $C$ | | | | | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

**2** Start to fill in the number of units you wish to send along each route, beginning at the north-west corner. Depot $W$ requires 11 lorry loads. This does not exhaust the stock of supplier $A$.

| | $W$ | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|---|
| $A$ | 11 | 3 | | | 14 |
| $B$ | | | | | 16 |
| $C$ | | | | | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

**3** The demand at $W$ has been met so move one square to the right and allocate $14 - 11 = 3$ units. The stock at $A$ is now exhausted. The demand at $X$ has not been met.

| | $W$ | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|---|
| $A$ | 11 | 3 | | | 14 |
| $B$ | | 12 | | | 16 |
| $C$ | | | | | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

As the stock at $A$ has been exhausted, move one square down and allocate the maximum possible number of units from supplier $B$ to depot $X$. In this case, $15 - 3 = 12$.

| | $W$ | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|---|
| $A$ | 11 | 3 | | | 14 |
| $B$ | | 12 | 4 | | 16 |
| $C$ | | | | | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

Now that the demand at $X$ has also been met, move one square to the right and use the remaining stock at $B$ to start to meet the demand at $Y$.

| | $W$ | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|---|
| $A$ | 11 | 3 | | | 14 |
| $B$ | | 12 | 4 | | 16 |
| $C$ | | | 10 | | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

The stock at $B$ is now exhausted (12 + 4 = 16) so move one square down and use the stock at $C$ to fulfil the remaining demand at $Y$.

| | $W$ | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|---|
| $A$ | 11 | 3 | | | 14 |
| $B$ | | 12 | 4 | | 16 |
| $C$ | | | 10 | 10 | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

Finally, move one square to the right and use the remaining stock at $C$ to meet the demand at $Z$.

This is the final table. All of the stock has been used and all of the demands met.

Use this table, together with the table showing costs, to work out the total cost of the solution.

**Problem-solving**

Number of occupied cells (routes used) in the table = number of supply points + number of demand points – 1.
In this case,
number of occupied cells (routes used) = 6
number of supply points = 3
number of demand points = 4
and 6 = 3 + 4 – 1.

Cost matrix

| | $W$ | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|---|
| $A$ | 180 | 110 | 130 | 290 | 14 |
| $B$ | 190 | 250 | 150 | 280 | 16 |
| $C$ | 240 | 270 | 190 | 120 | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

**Watch out** The cost matrix shows the cost of transporting **one unit**. Multiply the entry in each relevant position by the number of units being transported by that route. The relevant entries are shaded in this table, as these are the routes used in this initial solution.

The total cost of this solution is
$(11 \times 180) + (3 \times 110) + (12 \times 250)$
$+ (4 \times 150) + (10 \times 190) + (10 \times 120)$
= £9010

## Exercise 1A

**In questions 1 to 4**, the tables show the unit costs of transporting goods from supply points to demand points, the number of units required at each demand point and the number of units available at each supply point. In each case:

**a** use the north-west corner method to find the initial solution

**b** verify that, for each solution,
number of occupied cells = number of supply points + number of demand points – 1

**c** determine the cost of each initial solution.

A

1

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *A* | 150 | 213 | 222 | 32 |
| *B* | 175 | 204 | 218 | 44 |
| *C* | 188 | 198 | 246 | 34 |
| Demand | 28 | 45 | 37 | 110 |

2

| | *P* | *Q* | *R* | *S* | Supply |
|---|---|---|---|---|---|
| *A* | 27 | 33 | 34 | 41 | 54 |
| *B* | 31 | 29 | 37 | 30 | 67 |
| *C* | 40 | 32 | 28 | 35 | 29 |
| Demand | 21 | 32 | 51 | 46 | 150 |

3

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *A* | 17 | 24 | 19 | 123 |
| *B* | 15 | 21 | 25 | 143 |
| *C* | 19 | 22 | 18 | 84 |
| *D* | 20 | 27 | 16 | 150 |
| Demand | 200 | 100 | 200 | 500 |

4

| | *P* | *Q* | *R* | *S* | Supply |
|---|---|---|---|---|---|
| *A* | 56 | 86 | 80 | 61 | 134 |
| *B* | 59 | 76 | 78 | 65 | 203 |
| *C* | 62 | 70 | 57 | 67 | 176 |
| *D* | 60 | 68 | 75 | 71 | 187 |
| Demand | 175 | 175 | 175 | 175 | 700 |

## 1.2 Unbalanced problems and degenerate solutions

In the cost matrices given in the previous section, the total supply was always equal to the total demand. In real life this is rarely the case.

- **When total supply ≠ total demand, a transportation problem is unbalanced.**
  - **If total supply > total demand, you need to add a dummy demand point.**
  - **If total supply < total demand, you need to add a dummy supply point.**

**In each case the demand or supply at the dummy is chosen so that total demand is equal to total supply, and the transportation costs to/from the dummy location are all zero.**

**Hint** If units are assigned to a dummy demand point, they represent excess capacity. If a particular demand point is receiving units from a dummy supply point, they represent unmet demand.

### Example 3

Three outlets *A*, *B* and *C* are supplied by three suppliers *X*, *Y* and *Z*. The table shows the cost, in pounds, of transporting each unit, the number of units required at each outlet and the number of units available at each supplier.

| | *A* | *B* | *C* | Supply |
|---|---|---|---|---|
| *X* | 9 | 11 | 10 | 40 |
| *Y* | 10 | 8 | 12 | 60 |
| *Z* | 12 | 7 | 8 | 50 |
| Demand | 50 | 40 | 30 | |

**a** Explain why it is necessary to add a dummy demand point in order to solve this problem.

**b** Add a dummy demand point and appropriate costs to the table.

**c** Use the north-west corner method to obtain an initial solution.

A

a The total supply is 150, but the total demand is 120. A dummy is needed to absorb this excess, so that total supply equals total demand.

b

| | $A$ | $B$ | $C$ | $D$ | Supply |
|---|---|---|---|---|---|
| $X$ | 9 | 11 | 10 | 0 | 40 |
| $Y$ | 10 | 8 | 12 | 0 | 60 |
| $Z$ | 12 | 7 | 8 | 0 | 50 |
| Demand | 50 | 40 | 30 | 30 | 150 |

Add a dummy column, $D$, where the demand is 30 (the amount by which the supply exceeds the demand), and the transportation costs are zero (since there is no actual transporting done).
The problem is now **balanced**: the total supply is equal to the total demand.

c Units transported

| | $A$ | $B$ | $C$ | $D$ | Supply |
|---|---|---|---|---|---|
| $X$ | 40 | | | | 40 |
| $Y$ | 10 | 40 | 10 | | 60 |
| $Z$ | | | 20 | 30 | 50 |
| Demand | 50 | 40 | 30 | 30 | 150 |

## Example 4

The table shows the cost in pounds of transporting each unit to three retailers $A$, $B$ and $C$ from three suppliers $W$, $X$ and $Y$. It also shows the number of units required by each supplier and the number available at each outlet.

| | $A$ | $B$ | $C$ | Supply |
|---|---|---|---|---|
| $W$ | 8 | 9 | 11 | 55 |
| $X$ | 9 | 10 | 12 | 70 |
| $Y$ | 11 | 10 | 8 | 65 |
| **Demand** | 74 | 72 | 68 | |

**a** Use the north-west corner method to obtain an initial solution.

**b** State, with a reason, which retailer has not had their demand fully met.

a The total supply available is
$55 + 70 + 65 = 190$ units
The total demand is
$74 + 72 + 68 = 214$ units
The problem is unbalanced since
total supply $\neq$ total demand
A dummy supply row is needed before we can use the north-west corner method.

Check the total supply and total demand to determine whether the problem is balanced or unbalanced.

Supply $<$ demand, so you need to add a dummy **supply point**, and an extra **row**.

| | $A$ | $B$ | $C$ | Supply |
|---|---|---|---|---|
| $W$ | 8 | 9 | 11 | 55 |
| $X$ | 9 | 10 | 12 | 70 |
| $Y$ | 11 | 10 | 8 | 65 |
| $Z$ | 0 | 0 | 0 | 24 |
| Demand | 74 | 72 | 68 | |

The extra supply needed is 214 − 190 = 24. This is allocated to the dummy supply, $Z$. The costs in this row are all 0.

Units transported

| | $A$ | $B$ | $C$ | Supply |
|---|---|---|---|---|
| $W$ | 55 | | | 55 |
| $X$ | 19 | 51 | | 70 |
| $Y$ | | 21 | 44 | 65 |
| $Z$ | | | 24 | 24 |
| Demand | 74 | 72 | 68 | |

**b** Retailers $A$ and $B$ have their demands fully met but retailer $C$ has a shortfall of 24 units.

**Problem-solving**

The initial solution shows 24 units being transported from the dummy supply point. These units do not exist, so they represent unmet demand.

- **In a feasible solution to a transportation problem with $m$ rows and $n$ columns, if the number of cells used is less than $n + m - 1$, then the solution is degenerate.**

  **This will happen when an entry, other than the last, is made that satisfies the supply for a given row, and at the same time satisfies the demand for a given column.**

  **The algorithm requires that $n + m - 1$ cells are used in every solution, so a zero needs to be placed in a currently unused cell.**

## Example 5

The table shows the unit cost in pounds of transporting goods from supply points $W$, $X$, $Y$ and $Z$ to demand points $A$, $B$ and $C$. It also shows the number of units required at each demand point and the number of units available at each supply point.

| | $A$ | $B$ | $C$ | Supply |
|---|---|---|---|---|
| $W$ | 10 | 11 | 6 | 30 |
| $X$ | 4 | 5 | 9 | 20 |
| $Y$ | 3 | 8 | 7 | 35 |
| $Z$ | 11 | 10 | 9 | 35 |
| **Demand** | 30 | 40 | 50 | 120 |

**a** Demonstrate that the north-west corner method gives a degenerate solution and explain why it is degenerate.

**b** Adapt your solution to give a non-degenerate initial solution and state its cost.

A

Units transported

a

| | $A$ | $B$ | $C$ | Supply |
|---|---|---|---|---|
| $W$ | 30 | | | 30 |
| $X$ | | 20 | | 20 |
| $Y$ | | 20 | 15 | 35 |
| $Z$ | | | 35 | 35 |
| Demand | 30 | 40 | 50 | 120 |

There are 4 rows and 3 columns so a non-degenerate solution will use $4 + 3 - 1 = 6$ cells. This solution is degenerate since it fulfils all the supply and demand needs but only uses 5 cells $WA$, $XB$, $YB$, $YC$ and $ZC$.

Notice that there has been a diagonal 'move' from cell $WA$ to cell $XB$. Degenerate solutions can be avoided by not allowing diagonal moves.

b There are two possible initial solutions, depending on where you chose to place the zero.

| | $A$ | $B$ | $C$ | Supply |
|---|---|---|---|---|
| $W$ | 30 | | | 30 |
| $X$ | | | | 20 |
| $Y$ | | | | 35 |
| $Z$ | | | | 35 |
| Demand | 30 | 40 | 50 | 120 |

Start by placing the largest possible number in the north-west corner.

Either

| | $A$ | $B$ | $C$ | Supply |
|---|---|---|---|---|
| $W$ | 30 | 0 | | 30 |
| $X$ | | 20 | | 20 |
| $Y$ | | 20 | 15 | 35 |
| $Z$ | | | 35 | 35 |
| Demand | 30 | 40 | 50 | 120 |

Or

| | $A$ | $B$ | $C$ | Supply |
|---|---|---|---|---|
| $W$ | 30 | | | 30 |
| $X$ | 0 | 20 | | 20 |
| $Y$ | | 20 | 15 | 35 |
| $Z$ | | | 35 | 35 |
| Demand | 30 | 40 | 50 | 120 |

Having placed the 30 in the north-west corner, you now need to place the next number in the square to its right or in the square underneath.

Since both the supply and the demand are satisfied, this means that you will have to place a zero in one of these two cells.

Both have a cost of $(30 \times 10) + 0 + (20 \times 5) + (20 \times 8) + (15 \times 7) + (35 \times 9) = £980$

**Note** In fact the zero could be placed anywhere in the table, but it is convenient to 'stick to the rule' about restricting the movement to one square down or one square right.

## Exercise 1B

A E/P **1** Four sandwich shops $A$, $B$, $C$ and $D$ can be supplied with bread from three bakeries $X$, $Y$ and $Z$. The table shows the cost, in pence, of transporting one tray of bread from each supplier to each shop, the number of trays of bread required by each shop and the number of trays of bread that can be supplied by each bakery.

| | $A$ | $B$ | $C$ | $D$ | **Supply** |
|---|---|---|---|---|---|
| $X$ | 27 | 33 | 34 | 41 | 60 |
| $Y$ | 31 | 29 | 37 | 30 | 60 |
| $Z$ | 40 | 32 | 28 | 35 | 80 |
| **Demand** | 40 | 70 | 50 | 20 | |

**a** Explain why it is necessary to add a dummy demand point in order to solve this problem, and what this dummy point means in practical terms. **(1 mark)**

**b** Use the north-west corner method to determine an initial solution to this problem and the cost of this solution. **(2 marks)**

**2** A company needs to supply ready-mixed concrete from four depots $A$, $B$, $C$ and $D$ to four work sites $K$, $L$, $M$ and $N$. The number of loads that can be supplied from each depot and the number of loads required at each site are shown in the table, as well as the transportation cost per load from each depot to each work site.

| | $K$ | $L$ | $M$ | $N$ | **Supply** |
|---|---|---|---|---|---|
| $A$ | 35 | 46 | 62 | 80 | 20 |
| $B$ | 24 | 53 | 73 | 52 | 15 |
| $C$ | 67 | 61 | 50 | 65 | 20 |
| $D$ | 92 | 81 | 41 | 42 | 20 |
| **Demand** | 25 | 10 | 18 | 22 | |

**a** Explain what is meant by a degenerate solution.

**b** Demonstrate that the north-west corner method gives a degenerate solution.

**c** Adapt your solution to give a non-degenerate initial solution.

P **3** The table shows a balanced transportation problem.

| | $L$ | $M$ | $N$ | **Supply** |
|---|---|---|---|---|
| $P$ | 3 | 5 | 9 | 22 |
| $Q$ | 4 | 3 | 7 | $a$ |
| $R$ | 6 | 4 | 8 | 11 |
| $S$ | 8 | 2 | 5 | $b$ |
| **Demand** | 15 | 17 | 20 | |

The initial solution, given by the north-west corner method, is degenerate.

**a** Use this information to determine the values of $a$ and $b$.

**b** Hence write down the initial, degenerate solution given by the north-west corner method.

**c** Explain how to adapt the solution so that it is no longer degenerate.

A E/P **4** Three companies $A$, $B$ and $C$ supply tyres to three garages $P$, $Q$ and $R$. The table shows the cost in pounds of transporting each tyre from a company to a garage. It also shows the number of tyres available at each company and the number of tyres required at each garage for a period of one week.

| | $P$ | $Q$ | $R$ | Supply |
|---|---|---|---|---|
| $A$ | 4 | 5 | 3 | 28 |
| $B$ | 5 | 5 | 4 | 26 |
| $C$ | 4 | 6 | 5 | 31 |
| Demand | 24 | 30 | 45 | |

**a** Explain why a dummy supply point is needed. **(1 mark)**

**b** Use the north-west corner method to find an initial solution using a dummy supply, $D$. Explain clearly how you avoided a degenerate solution. **(2 marks)**

**c** Interpret the value in row $D$ of your initial solution. **(1 mark)**

## 1.3 Finding an improved solution

So far, you have only found initial solutions to the transportation problem. You now need to consider how the solution can be improved in order to reduce the costs.

To find an **improved solution**, you need to:

1. Use the non-empty cells to find the **shadow costs**. → Examples 6, 7
2. Use the shadow costs and the empty cells to find **improvement indices**. → Examples 8, 9
3. Use the improvement indices and the **stepping-stone method** to find an improved solution. → Example 10, Section 1.4

Consider the initial solution found in Example 2. The costs associated with the routes used in the solution have been highlighted.

**Cost matrix**

| | $W$ | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|---|
| $A$ | 180 | 110 | 130 | 290 | 14 |
| $B$ | 190 | 250 | 150 | 280 | 16 |
| $C$ | 240 | 270 | 190 | 120 | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

**Initial solution**

| | $W$ | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|---|
| $A$ | 11 | 3 | | | 14 |
| $B$ | | 12 | 4 | | 16 |
| $C$ | | | 10 | 10 | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

For a given route, you can define each cost in terms of the portion of the cost due to supply costs, and the portion of the cost due to demand costs. These costs are called shadow costs.

- **Transportation costs are made up of two components, one associated with the source and one with the destination. These costs of using that route are called shadow costs.**

A The cost of transporting a unit between two points is the sum of these costs. For example:

S($A$) + D($X$) = 110 — This is the cost of transporting a unit from $A$ to $X$.

S($C$) + D($Z$) = 120

**Notation** The shadow cost associated with a given supply point, $A$, is written S($A$) or $R_A$. The shadow cost associated with a given demand point, $X$, is written D($X$) or $K_X$.

For any given solution, you will have more unknowns than equations. In the above example, there are 6 equations based on the shadow costs for the 6 routes that are used in that solution, but there are 7 unknowns (the shadow costs for each source and destination). However, by setting one value arbitrarily equal to 0, you can solve to find the other shadow costs.

**Watch out** You only find shadow costs associated with a **particular solution**, so you will only ever consider $n + m - 1$ equations.

To find the **shadow costs**, follow these steps:

1. Start with the north-west corner, and set the cost linked with its source to zero.
2. Move along the row to any other non-empty squares. Set the cost linked with these destinations equal to the total transportation cost for that route (since the source cost for the first row is 0).
3. When all possible destination costs for that row have been established, go to the start of the next row.
4. Move along this row to any non-empty squares and use the destination costs found earlier, to establish the source cost for the row. Once that has been done, find any further unknown destination costs.
5. Repeat steps 3 and 4 until all source and destination costs have been found.

## Example 6

Calculate the shadow costs given by the initial solution of the problem given in Example 2 shown in the table below.

| | Depot $W$ | Depot $X$ | Depot $Y$ | Depot $Z$ | Stock |
|---|---|---|---|---|---|
| **Supplier $A$** | 180 | 110 | 130 | 290 | 14 |
| **Supplier $B$** | 190 | 250 | 150 | 280 | 16 |
| **Supplier $C$** | 240 | 270 | 190 | 120 | 20 |
| **Demand** | 11 | 15 | 14 | 10 | 50 |

A

Initial solution (see page 5) was

| | $W$ | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|---|
| $A$ | 11 | 3 | | | 14 |
| $B$ | | 12 | 4 | | 16 |
| $C$ | | | 10 | 10 | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

**Online** Explore how to calculate shadow costs using GeoGebra.

Fill in the **costs** of the routes being used – **the non-empty squares.**

| | Depot $W$ | Depot $X$ | Depot $Y$ | Depot $Z$ |
|---|---|---|---|---|
| Supplier $A$ | 180 | 110 | | |
| Supplier $B$ | | 250 | 150 | |
| Supplier $C$ | | | 190 | 120 |

Remember to use the cost values, not the number of items currently being transported along that route.

**Note** The Demand row and the Stock column are not needed when finding the shadow costs.

Putting S($A$) to zero, from row 1 we get D($W$) = 180 and D($X$) = 110.

| Shadow costs | | 180 | 110 | | |
|---|---|---|---|---|---|
| | | Depot $W$ | Depot $X$ | Depot $Y$ | Depot $Z$ |
| 0 | Supplier $A$ | 180 | 110 | | |
| | Supplier $B$ | | 250 | 150 | |
| | Supplier $C$ | | | 190 | 120 |

Put S($A$) = 0 arbitrarily and then solve the equations S($A$) + D($W$) = 180 and S($A$) + D($X$) = 110.

Now move to row 2:
We know that D($X$) = 110, so S($B$) = 140 and hence D($Y$) = 10.

| Shadow costs | | 180 | 110 | 10 | |
|---|---|---|---|---|---|
| | | Depot $W$ | Depot $X$ | Depot $Y$ | Depot $Z$ |
| 0 | Supplier $A$ | 180 | 110 | | |
| 140 | Supplier $B$ | | 250 | 150 | |
| | Supplier $C$ | | | 190 | 120 |

Solve S($B$) + D($X$) = 250 to get S($B$) = 140, and use this together with the equation S($B$) + D($Y$) = 150 to get D($Y$) = 10.

Move to row 3:
We know that D($Y$) = 10, so S($C$) = 180 and hence D($Z$) = −60.

| Shadow costs | | 180 | 110 | 10 | −60 |
|---|---|---|---|---|---|
| | | Depot $W$ | Depot $X$ | Depot $Y$ | Depot $Z$ |
| 0 | Supplier $A$ | 180 | 110 | | |
| 140 | Supplier $B$ | | 250 | 150 | |
| 180 | Supplier $C$ | | | 190 | 120 |

Solve S($C$) + D($Y$) = 190 to get S($C$) = 180, and then solve S($C$) + D($Z$) = 120 to find D($Z$).

We have now found all source and all destination shadow costs:
S($A$) = 0, S($B$) = 140, S($C$) = 180,
D($W$) = 180, D($X$) = 110, D($Y$) = 10,
D($Z$) = −60.

**Problem-solving**

Because you set S($A$) = 0, you are in fact finding relative costs, not actual costs. This can give rise to negative values, as seen in this table. These relative costs are suitable for the purposes of implementing the algorithm.

## Example 7

A

Calculate the shadow costs given by the initial solution of the problem given in Example 3 shown in the table below.

| | *A* | *B* | *C* | *D* | Supply |
|---|---|---|---|---|---|
| *X* | 9 | 11 | 10 | 0 | 40 |
| *Y* | 10 | 8 | 12 | 0 | 60 |
| *Z* | 12 | 7 | 8 | 0 | 50 |
| **Demand** | 50 | 40 | 30 | 30 | 150 |

| | *A* | *B* | *C* | *D* | Supply |
|---|---|---|---|---|---|
| *X* | 40 | | | | 40 |
| *Y* | 10 | 40 | 10 | | 60 |
| *Z* | | | 20 | 30 | 50 |
| Demand | 50 | 40 | 30 | 30 | |

The initial solution shows the number of items being transported, not the costs.

Using the costs associated with the routes used in the initial solution:

| Shadow costs | | | | | |
|---|---|---|---|---|---|
| | | *A* | *B* | *C* | *D* |
| | *X* | 9 | | | |
| | *Y* | 10 | 8 | 12 | |
| | *Z* | | | 8 | 0 |

Add a new row and column to your table to show shadow costs. Enter values as you calculate them, then use these values to calculate subsequent shadow costs.

Arbitrarily assign S(*X*) = 0:

| Shadow costs | | | | | |
|---|---|---|---|---|---|
| | | *A* | *B* | *C* | *D* |
| 0 | *X* | 9 | | | |
| | *Y* | 10 | 8 | 12 | |
| | *Z* | | | 8 | 0 |

Use this to work out the shadow cost for D(*A*):

| Shadow costs | | 9 | | | |
|---|---|---|---|---|---|
| | | *A* | *B* | *C* | *D* |
| 0 | *X* | 9 | | | |
| | *Y* | 10 | 8 | 12 | |
| | *Z* | | | 8 | 0 |

A

Use this to work out the shadow cost for S($Y$):

| Shadow costs | | 9 | | | |
|---|---|---|---|---|---|
| | | $A$ | $B$ | $C$ | $D$ |
| 0 | $X$ | 9 | | | |
| 1 | $Y$ | 10 | 8 | 12 | |
| | $Z$ | | | 8 | 0 |

From the cost matrix, S($Y$) + D($A$) = 10. You have already found D($A$) = 9, so S($Y$) = 1.

Then work out the shadow costs for D($B$) and D($C$):

| Shadow costs | | 9 | 7 | 11 | |
|---|---|---|---|---|---|
| | | $A$ | $B$ | $C$ | $D$ |
| 0 | $X$ | 9 | | | |
| 1 | $Y$ | 10 | 8 | 12 | |
| | $Z$ | | | 8 | 0 |

Use these to work out the shadow cost for S($Z$):

| Shadow costs | | 9 | 7 | 11 | |
|---|---|---|---|---|---|
| | | $A$ | $B$ | $C$ | $D$ |
| 0 | $X$ | 9 | | | |
| 1 | $Y$ | 10 | 8 | 12 | |
| −3 | $Z$ | | | 8 | 0 |

Then work out the shadow cost for D($D$):

| Shadow costs | | 9 | 7 | 11 | 3 |
|---|---|---|---|---|---|
| | | $A$ | $B$ | $C$ | $D$ |
| 0 | $X$ | 9 | | | |
| 1 | $Y$ | 10 | 8 | 12 | |
| −3 | $Z$ | | | 8 | 0 |

You do not have to show each stage of the table in your exam. Just this final table of shadow costs would be sufficient.

It may be possible to reduce the cost of the initial solution by introducing a route that is not currently in use. You consider each unused route in turn and calculate the reduction in cost which would be made by sending one unit along that route. This is called the **improvement index**.

- **The improvement index in sending a unit from a source $P$ to a demand point $Q$ is found by subtracting the source cost S($P$) and destination cost D($Q$) from the stated cost of transporting one unit along that route C($PQ$):**

  **Improvement index for $PQ = I_{PQ} = C(PQ) - S(P) - D(Q)$**

- **The route with the most negative improvement index will be introduced into the solution.**
- **The cell corresponding to the value with the most negative improvement index becomes the entering cell (or entering square or entering route) and the route it replaces is referred to as the exiting cell (or exiting square or exiting route).**

- **If there are two equal potential entering cells you may choose either. Similarly, if there are two equal exiting cells, you may select either.**
- **If there are no negative improvement indices the solution is optimal.**

**Online** Explore how to calculate improvement indices using GeoGebra.

## Example 8

| Shadow costs | | 180 | 110 | 10 | −60 |
|---|---|---|---|---|---|
| | | **Depot *W*** | **Depot *X*** | **Depot *Y*** | **Depot *Z*** |
| 0 | **Supplier *A*** | 180 | 110 | 130 | 290 |
| 140 | **Supplier *B*** | 190 | 250 | 150 | 280 |
| 180 | **Supplier *C*** | 240 | 270 | 190 | 120 |

Use the shadow costs found in Example 6, and shown in the table above, to calculate improvement indices, and use these to identify the entering cell.

Focus on the routes not currently being used: $BW$, $CW$, $CX$, $AY$, $AZ$ and $BZ$.
The known shadow costs are:
$S(A) = 0$, $S(B) = 140$, $S(C) = 180$, $D(W) = 180$, $D(X) = 110$, $D(Y) = 10$, $D(Z) = -60$
Improvement index for $BW = I_{BW} = C(BW) - S(B) - D(W) = 190 - 140 - 180 = -130$
Improvement index for $CW = I_{CW} = 240 - 180 - 180 = -120$
Improvement index for $CX = I_{CX} = 270 - 180 - 110 = -20$
Improvement index for $AY = I_{AY} = 130 - 0 - 10 = 120$
Improvement index for $AZ = I_{AZ} = 290 - 0 - (-60) = 350$
Improvement index for $BZ = I_{BZ} = 280 - 140 - (-60) = 200$
The entering cell is therefore $BW$, since this has the most negative improvement index.

A more concise approach is to write the improvement indices in a table.
The routes already in use are marked with a cross.

| **Shadow costs** | | 180 | 110 | 10 | −60 |
|---|---|---|---|---|---|
| | | ***W*** | ***X*** | ***Y*** | ***Z*** |
| 0 | ***A*** | × | × | | |
| 140 | ***B*** | | × | × | |
| 180 | ***C*** | | | × | × |

The calculations are carried out exactly as above, but without all the working, and the results are entered directly into the table.

| **Shadow costs** | | 180 | 110 | 10 | −60 |
|---|---|---|---|---|---|
| | | ***W*** | ***X*** | ***Y*** | ***Z*** |
| 0 | ***A*** | × | × | 120 | 350 |
| 140 | ***B*** | −130 | × | × | 200 |
| 180 | ***C*** | −120 | −20 | × | × |

As before, the entering cell is seen to be $BW$ as it has the most negative value.

**Note** You may be given several copies of a blank table in your exam. You don't have to use all of them, just the ones that you need.

The table shown below is typical of the type provided.

| | *W* | *X* | *Y* | *Z* |
|---|---|---|---|---|
| *A* | | | | |
| *B* | | | | |
| *C* | | | | |

You will be expected to write the shadow costs in the appropriate places to give a final response like the one below.

| | | 180 | 110 | 10 | −60 |
|---|---|---|---|---|---|
| | | *W* | *X* | *Y* | *Z* |
| 0 | *A* | × | × | 120 | 350 |
| 140 | *B* | −130 | × | × | 200 |
| 180 | *C* | −120 | −20 | × | × |

## Example 9

**a** Use the north-west corner method to find an initial solution to the transportation problem shown in the table.

**b** Find the shadow costs and improvement indices.

**c** Hence determine whether the solution is optimal.

| | *X* | *Y* | *Z* | Supply |
|---|---|---|---|---|
| *A* | 11 | 12 | 17 | 11 |
| *B* | 13 | 10 | 13 | 15 |
| *C* | 15 | 18 | 9 | 14 |
| Demand | 10 | 15 | 15 | |

a Units transported

| | *X* | *Y* | *Z* | Supply |
|---|---|---|---|---|
| *A* | 10 | 1 | | 11 |
| *B* | | 14 | 1 | 15 |
| *C* | | | 14 | 14 |
| Demand | 10 | 15 | 15 | |

b Shadow costs

| | | 11 | 12 | 15 |
|---|---|---|---|---|
| | | *X* | *Y* | *Z* |
| 0 | *A* | 11 | 12 | |
| −2 | *B* | | 10 | 13 |
| −6 | *C* | | | 9 |

Use the costs corresponding to the route found in part **a** to determine the shadow costs.

Improvement indices

| | | 11 | 12 | 15 |
|---|---|---|---|---|
| | | *X* | *Y* | *Z* |
| 0 | *A* | × | × | 2 |
| −2 | *B* | 4 | × | × |
| −6 | *C* | 10 | 12 | × |

Replace each cost corresponding to the current route with a cross.
Write the improvement indices in the remaining cells.

c There are no negative improvement indices, so the solution is optimal.

## Exercise 1C

**In questions 1 to 4,** start with the initial, north-west corner solutions found in questions 1 to 4 of Exercise 1A. In each case use the initial solution, and the original cost matrix, shown below, to find:

**a** the shadow costs

**b** the improvement indices

**c** the entering cell, if appropriate.

**1**

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *A* | 150 | 213 | 222 | 32 |
| *B* | 175 | 204 | 218 | 44 |
| *C* | 188 | 198 | 246 | 34 |
| Demand | 28 | 45 | 37 | |

**2**

| | *P* | *Q* | *R* | *S* | Supply |
|---|---|---|---|---|---|
| *A* | 27 | 33 | 34 | 41 | 54 |
| *B* | 31 | 29 | 37 | 30 | 67 |
| *C* | 40 | 32 | 28 | 35 | 29 |
| Demand | 21 | 32 | 51 | 46 | |

**3**

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *A* | 17 | 24 | 19 | 123 |
| *B* | 15 | 21 | 25 | 143 |
| *C* | 19 | 22 | 18 | 84 |
| *D* | 20 | 27 | 16 | 150 |
| Demand | 200 | 100 | 200 | |

**4**

| | *P* | *Q* | *R* | *S* | Supply |
|---|---|---|---|---|---|
| *A* | 56 | 86 | 80 | 61 | 134 |
| *B* | 59 | 76 | 78 | 65 | 203 |
| *C* | 62 | 70 | 57 | 67 | 176 |
| *D* | 60 | 68 | 75 | 71 | 187 |
| Demand | 175 | 175 | 175 | 175 | |

E/P **5** The table below shows the cost in pounds of transporting one unit of stock from each of three supply points *A*, *B* and *C* to three demand points *X*, *Y* and *Z*.

| | *X* | *Y* | *Z* | Supply |
|---|---|---|---|---|
| *A* | 39 | 54 | 47 | 76 |
| *B* | 48 | 55 | 61 | 68 |
| *C* | 52 | 44 | 58 | 60 |
| Demand | 83 | 57 | 64 | |

**a** This is a **balanced problem**. Explain what this means. **(1 mark)**

**b** Use the north-west corner method to find an initial solution to this transportation problem. **(1 mark)**

**c** Show that there are some negative improvement indices and interpret this result. **(2 marks)**

## 1.4 The stepping-stone method

Once you have determined that an initial solution has a negative improvement index, and identified the entering cell, you can use an iterative process to obtain an improved solution.

In Example 8 you discovered that the most negative improvement index was $I_{BW}$, with a value of −130. This means that every time you send a unit along $BW$ you save a cost of 130. Therefore you want to send as many units as possible along this new route. You have to be careful, however, not to exceed the stock or the demand. To ensure this, you go through a sequence of adjustments, called the **stepping-stone method**.

You are looking therefore for a cycle of adjustments, where you increase the value in one cell and then decrease the value in the next cell, then increase the value in the next, and so on.

**The stepping-stone method**

**1** Create the cycle of adjustments. The two basic rules are:

**a** Within any row and any column there can only be one increasing cell and one decreasing cell.

**b** Apart from the entering cell, adjustments are only made to non-empty cells.

**2** Once the cycle of adjustments has been found you transfer the maximum number of units through this cycle. This will be equal to the smallest number in the decreasing cells (since you may not have negative units being transported).

**3** You then adjust the solution to incorporate this improvement.

**Hint** A popular mind picture is that you are using the cells as 'stepping-stones', placing one foot on each, and alternately putting down your left foot (increasing) then right foot (decreasing) as you journey around the table.

### Example 10

In Example 2, the table of costs was

| | Depot $W$ | Depot $X$ | Depot $Y$ | Depot $Z$ | Stock |
|---|---|---|---|---|---|
| Supplier $A$ | 180 | 110 | 130 | 290 | 14 |
| Supplier $B$ | 190 | 250 | 150 | 280 | 16 |
| Supplier $C$ | 240 | 270 | 190 | 120 | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

and the initial solution was

| | $W$ | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|---|
| $A$ | 11 | 3 | | | 14 |
| $B$ | | 12 | 4 | | 16 |
| $C$ | | | 10 | 10 | 20 |
| Demand | 11 | 15 | 14 | 10 | 50 |

The most negative improvement index was −130 in cell $BW$. ← Example 8

at a cost of £9010.

Obtain an improved solution and find the improved cost.

Use $BW$ as the entering cell. This gave the most negative improvement index, so $BW$ will be an increasing cell. Enter a value of $\theta$ into this cell.

$\theta$ is a non-negative value representing the number of units transported via the entering cell. You will later choose the largest possible value of $\theta$ that allows you to maintain the supply and demand constraints in other cells.

| | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|
| $A$ | 11 | 3 | | |
| $B$ | $\theta$ | 12 | 4 | |
| $C$ | | | 10 | 10 |

The adjustments can be made without keeping the Demand row and the Stock column.

In order to keep the demand at $W$ correct, we must decrease the entry at $AW$, so $AW$ will be a decreasing cell.

| | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|
| $A$ | $11 - \theta$ | 3 | | |
| $B$ | $\theta$ | 12 | 4 | |
| $C$ | | | 10 | 10 |

In order to keep the stock at $A$ correct, we must increase the entry at $AX$, so $AX$ will be an increasing cell.

| | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|
| $A$ | $11 - \theta$ | $3 + \theta$ | | |
| $B$ | $\theta$ | 12 | 4 | |
| $C$ | | | 10 | 10 |

**Online** Explore how to obtain an improved solution with the stepping stone method using GeoGebra.

In order to keep the demand at $X$ correct, we must decrease the entry at $BX$, so $BX$ will be a decreasing cell.

| | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|
| $A$ | $11 - \theta$ | $3 + \theta$ | | |
| $B$ | $\theta$ | $12 - \theta$ | 4 | |
| $C$ | | | 10 | 10 |

This is as far as you can go with adjustments since the top two rows both have an increasing and a decreasing cell.

A

Choose the largest possible value for $\theta$ that doesn't introduce any negative entries.

The largest possible value of $\theta$ is 11, so replace $\theta$ by 11 in the table:

| | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|
| $A$ | 11 – 11 | 3 + 11 | | |
| $B$ | 11 | 12 – 11 | 4 | |
| $C$ | | | 10 | 10 |

This gives the following improved solution.

| | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|
| $A$ | | 14 | | |
| $B$ | 11 | 1 | 4 | |
| $C$ | | | 10 | 10 |

Looking at the table of transportation costs, this solution has a cost of £7580.

**Note** All of the stages are shown here in separate tables to make the method clear but, in your exam, you can carry out all of the steps in a single table.

**Watch out** The table needs to show a feasible solution, so you can't have any negative entries. But the choice of $\theta$ will always reduce at least one other entry to 0.

**Problem-solving**

As a double check, it is always true that

New cost = cost of former solution + improvement index × $\theta$

In this case, 7580 = 9010 + (–130) × 11

In the above example, the choice of $\theta$ means that the number of units being transported via $AW$ is 0. Your solution needs to have exactly $n + m - 1$ entries in the table, so don't write 0 in $AW$. You should **leave it blank** instead.

- **One entry that has been reduced to 0 by the stepping-stone method is called the exiting cell, and should be left blank in the improved solution.**
- **At each iteration you create one entering cell and one exiting cell.**
- **To find an optimal solution, continue to calculate new shadow costs and improvement indices and then apply the stepping-stone method. Repeat this iteration until all the improvement indices are non-negative.**

**Watch out** If two cells are reduced to 0 then choose one as the exiting cell, and leave the other 0 in the table. You should always check that your solution has $n + m - 1$ entries.

## Example 11

Find an optimal solution for Example 10.

**Second iteration**

Find the new shadow costs:

| | | 50 | 110 | 10 | –60 |
|---|---|---|---|---|---|
| | | $W$ | $X$ | $Y$ | $Z$ |
| 0 | $A$ | | 110 | | |
| 140 | $B$ | 190 | 250 | 150 | |
| 180 | $C$ | | | 190 | 120 |

Find the new improvement indices:

| | | 50 | 110 | 10 | −60 |
|---|---|---|---|---|---|
| | | **W** | **X** | **Y** | **Z** |
| 0 | **A** | 130 | × | 120 | 350 |
| 140 | **B** | × | × | × | 200 |
| 180 | **C** | 10 | −20 | × | × |

So the new entering cell is $CX$, since this has the most negative improvement index.

Applying the stepping-stone method gives

| | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|
| $A$ | | 14 | | |
| $B$ | 11 | $1-\theta$ | $4+\theta$ | |
| $C$ | | $\theta$ | $10-\theta$ | 10 |

Looking at cells $BX$ and $CY$ we see that the greatest value for $\theta$ is 1.

The new exiting cell will be $BX$, $\theta = 1$ and we get

| | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|
| $A$ | | 14 | | |
| $B$ | 11 | | 5 | |
| $C$ | | 1 | 9 | 10 |

The new cost is £7560.

Checking, 7580 + (−20) × 1 = 7560

**Third iteration**

New shadow costs:

| | | 70 | 110 | 30 | −40 |
|---|---|---|---|---|---|
| | | **W** | **X** | **Y** | **Z** |
| 0 | **A** | | 110 | | |
| 120 | **B** | 190 | | 150 | |
| 160 | **C** | | 270 | 190 | 120 |

New improvement indices:

| | | 70 | 110 | 30 | −40 |
|---|---|---|---|---|---|
| | | **W** | **X** | **Y** | **Z** |
| 0 | **A** | 110 | × | 100 | 330 |
| 120 | **B** | × | 20 | × | 200 |
| 160 | **C** | 10 | × | × | × |

There are no negative improvement indices so this solution is optimal.

The solution is 14 units $A$ to $X$
11 units $B$ to $W$
5 units $B$ to $Y$
1 unit $C$ to $X$
9 units $C$ to $Y$
10 units $C$ to $Z$

At this point, if there is an improvement index of 0, this would indicate that there is an alternative optimal solution. To find it, use the cell with the zero improvement index as the entering cell.
→ Mixed exercise, Q2

Some stepping-stone routes are not rectangles and some $\theta$ values are not immediately apparent.

## Example 12

The table shows the unit cost, in pounds, of transporting goods from each of three warehouses $A$, $B$ and $C$ to each of three supermarkets $X$, $Y$ and $Z$. It also shows the stock at each warehouse and the demand at each supermarket.

| | Supermarket $X$ | Supermarket $Y$ | Supermarket $Z$ | Stock |
|---|---|---|---|---|
| **Warehouse $A$** | 24 | 22 | 28 | 13 |
| **Warehouse $B$** | 26 | 26 | 14 | 11 |
| **Warehouse $C$** | 20 | 22 | 20 | 12 |
| **Demand** | 10 | 13 | 13 | |

Solve the transportation problem shown in the table. Use the north-west corner method to obtain an initial solution. You must state your shadow costs, improvement indices, stepping-stone routes, $\theta$ values, entering cells and exiting cells. You must state the initial cost and the improved cost after each iteration.

Supply = demand = 36, so a dummy is not needed.

Check the problem is balanced.

The north-west corner method gives the following initial solution.

| | $X$ | $Y$ | $Z$ | Stock |
|---|---|---|---|---|
| $A$ | 10 | 3 | | 13 |
| $B$ | | 10 | 1 | 11 |
| $C$ | | | 12 | 12 |
| Demand | 10 | 13 | 13 | |

The cost of the initial solution is £820.

**First iteration**

Shadow costs:

| | | 24 | 22 | 10 |
|---|---|---|---|---|
| | | $X$ | $Y$ | $Z$ |
| 0 | $A$ | 24 | 22 | |
| 4 | $B$ | | 26 | 14 |
| 10 | $C$ | | | 20 |

A

Improvement indices:

| | | 24 | 22 | 10 |
|---|---|---|---|---|
| | | *X* | *Y* | *Z* |
| 0 | *A* | × | × | 18 |
| 4 | *B* | −2 | × | × |
| 10 | *C* | −14 | −10 | × |

Use *CX* as the entering cell, since it has the most negative improvement index.

| | *X* | *Y* | *Z* |
|---|---|---|---|
| *A* | $10 - \theta$ | $3 + \theta$ | |
| *B* | | $10 - \theta$ | $1 + \theta$ |
| *C* | $\theta$ | | $12 - \theta$ |

The maximum value of $\theta$ is 10.
Either cell *AX* or cell *BY* can be the exiting cell, since both of these will go to zero.
Choose *AX* to be the exiting cell. The other will have a numerical value of zero.
The improved solution is:

| | *X* | *Y* | *Z* | Stock |
|---|---|---|---|---|
| *A* | | 13 | | 13 |
| *B* | | 0 | 11 | 11 |
| *C* | 10 | | 2 | 12 |
| Demand | 10 | 13 | 13 | |

The cost is now £680.

**Second iteration**

Check for optimality by calculating improvement indices.
The second set of shadow costs are:

| | | 10 | 22 | 10 |
|---|---|---|---|---|
| | | *X* | *Y* | *Z* |
| 0 | *A* | | 22 | |
| 4 | *B* | | 26 | 14 |
| 10 | *C* | 20 | | 20 |

The improvement indices are:

| | | 10 | 22 | 10 |
|---|---|---|---|---|
| | | *X* | *Y* | *Z* |
| 0 | *A* | 14 | × | 18 |
| 4 | *B* | 12 | × | × |
| 10 | *C* | × | −10 | × |

**Watch out** This stepping-stone route is quite complicated. Take a minute or two to check how it has been created. Start by putting $\theta$ in *CX*, then to correct the demand of *X*, subtract $\theta$ from cell *AX*, then to correct the supply in *A*, add $\theta$ to cell *AY* and so on, finishing at cell *CZ*.

**Watch out** Many candidates fail to understand the difference between an empty cell and one with a zero entry. Zero is a number, just like 4 and 'counts' towards our $m + n - 1$ entries in the table. An empty cell has no number in it.

A

The negative improvement index indicates that the algorithm is not complete. Apply the stepping-stone method again, using $CY$ as the entering cell.

| | $X$ | $Y$ | $Z$ |
|---|---|---|---|
| $A$ | | 13 | |
| $B$ | | $0 - \theta$ | $11 + \theta$ |
| $C$ | 10 | $\theta$ | $2 - \theta$ |

It can be seen from cell $BY$ that the maximum value of $\theta$ is 0.
The entering cell is $CY$ and will have an entry of 0.
The exiting cell is $BY$ and it will now be empty.
The improved solution is:

| | $X$ | $Y$ | $Z$ |
|---|---|---|---|
| $A$ | | 13 | |
| $B$ | | | 11 |
| $C$ | 10 | 0 | 2 |

The cost is unchanged at £680.

**Third iteration**

Shadow costs:

| | | 20 | 22 | 20 |
|---|---|---|---|---|
| | | $X$ | $Y$ | $Z$ |
| 0 | $A$ | | 22 | |
| −6 | $B$ | | | 14 |
| 0 | $C$ | 20 | 22 | 20 |

Improvement indices:

| | | 20 | 22 | 20 |
|---|---|---|---|---|
| | | $X$ | $Y$ | $Z$ |
| 0 | $A$ | 4 | × | 8 |
| −6 | $B$ | 12 | 10 | × |
| 0 | $C$ | × | × | × |

All the improvement indices are non-negative, so the solution is optimal.

| | $X$ | $Y$ | $Z$ |
|---|---|---|---|
| $A$ | | 13 | |
| $B$ | | | 11 |
| $C$ | 10 | 0 | 2 |

**Watch out** The existence of a negative improvement index does not necessarily mean that the solution is non-optimal. However, it means that you cannot yet determine from the algorithm that the solution is optimal. To show optimality, you need to continue applying the algorithm until there are no negative improvement indices.

This looks odd, but the algorithm must be followed. $\theta$ cannot be negative but it can be any non-negative number, including 0.

Check this improved solution for optimality by calculating the shadow costs and improvement indices.

The optimal solution is to send:
13 units from $A$ to $Y$
11 units from $B$ to $Z$
10 units from $C$ to $X$
2 units from $C$ to $Z$
at a cost of £680.

**Problem-solving**

Notice that the solution has the same value as that given by the previous iteration. This shows that the previous solution was in fact optimal, even though there was a negative improvement index.

## Exercise 1D

**In questions 1 to 3**, complete your solutions to the transportation problems from questions 1, 2 and 4 in Exercise 1C. You should demonstrate that your solution is optimal and calculate its total cost.

**1**

| | $P$ | $Q$ | $R$ | Supply |
|---|---|---|---|---|
| $A$ | 150 | 213 | 222 | 32 |
| $B$ | 175 | 204 | 218 | 44 |
| $C$ | 188 | 198 | 246 | 34 |
| Demand | 28 | 45 | 37 | |

**2**

| | $P$ | $Q$ | $R$ | $S$ | Supply |
|---|---|---|---|---|---|
| $A$ | 27 | 33 | 34 | 41 | 54 |
| $B$ | 31 | 29 | 37 | 30 | 67 |
| $C$ | 40 | 32 | 28 | 35 | 29 |
| Demand | 21 | 32 | 51 | 46 | |

**3**

| | $P$ | $Q$ | $R$ | $S$ | Supply |
|---|---|---|---|---|---|
| $A$ | 56 | 86 | 80 | 61 | 134 |
| $B$ | 59 | 76 | 78 | 65 | 203 |
| $C$ | 62 | 70 | 57 | 67 | 176 |
| $D$ | 60 | 68 | 75 | 71 | 187 |
| Demand | 175 | 175 | 175 | 175 | |

**Watch out** This solution will require a number of iterations. Keep going until you obtain no negative improvement indices.

**4** The table on the right shows the unit cost, in pounds, of transporting goods from each of three warehouses $A$, $B$ and $C$ to each of two supermarkets $P$ and $Q$. It also shows the stock at each warehouse and the demand at each supermarket.

| | $P$ | $Q$ | Supply |
|---|---|---|---|
| $A$ | 2 | 6 | 3 |
| $B$ | 2 | 7 | 5 |
| $C$ | 6 | 9 | 2 |
| Demand | 6 | 4 | |

Solve the transportation problem shown in the table. Use the north-west corner method to obtain an initial solution. You must state your shadow costs, improvement indices, stepping-stone routes, $\theta$ values, entering cells and exiting cells. You must state the initial cost and the improved cost after each iteration. **(10 marks)**

A E **5** The table below shows the cost, in pounds, of transporting one pallet of stock from each of three suppliers $X$, $Y$ and $Z$ to three warehouses $A$, $B$ and $C$. It also shows the stock held by each supplier and the stock required at each warehouse. A minimum cost solution is required.

| | $A$ | $B$ | $C$ | Supply |
|---|---|---|---|---|
| $X$ | 16 | 21 | 15 | 17 |
| $Y$ | 23 | 22 | 19 | 14 |
| $Z$ | 18 | 24 | 16 | 14 |
| **Demand** | 12 | 15 | 18 | |

**a** Show that the problem is balanced. **(1 mark)**

**b** Use the north-west corner method to find an initial solution. **(2 marks)**

**c** Calculate the shadow costs and improvement indices and explain why you cannot determine whether the initial solution is optimal. **(3 marks)**

**d** Use the stepping-stone method to find an improved solution. You must make your method clear by showing improvement indices, routes, entering cells and exiting cells. **(3 marks)**

**e** Verify that your improved solution is optimal and state the cost. **(2 marks)**

E/P **6** The table shows the cost of transporting one unit of stock from each of three supply points $A$, $B$ and $C$, to each of three demand points $D$, $E$ and $F$. It also shows the stock held at each supply point and the stock required at each demand point. A minimal cost solution is required.

| | $D$ | $E$ | $F$ | Supply |
|---|---|---|---|---|
| $A$ | 24 | 32 | 25 | 26 |
| $B$ | 27 | 21 | 28 | 23 |
| $C$ | 19 | 26 | 30 | 24 |
| **Demand** | 22 | 16 | 18 | |

**a** Explain why it is necessary to add a dummy demand point. **(1 mark)**

**b** Introduce a dummy demand point, $G$, and use the north-west corner method to find an initial solution. **(3 marks)**

**c** Calculate the shadow costs and improvement indices. **(3 marks)**

**d** Use the stepping-stone method once to obtain an improved solution. You must make your routes clear and state your entering and exiting cell. **(3 marks)**

E/P **7** The table shows the cost of transporting one unit of stock from each of three warehouses $A$, $B$ and $C$, to each of three outlets $X$, $Y$ and $Z$. It also shows the stock held at each warehouse and the stock required at each outlet. A minimal cost solution is required.

| | $X$ | $Y$ | $Z$ | Supply |
|---|---|---|---|---|
| $A$ | 18 | 25 | 21 | 30 |
| $B$ | 26 | 19 | 27 | 34 |
| $C$ | 20 | 28 | 24 | 35 |
| **Demand** | 35 | 37 | 42 | |

**a** Explain why a dummy supply point must be used. **(1 mark)**

A

**b** Use the transportation algorithm to find an optimal solution. You must show:
- an initial feasible solution
- shadow costs
- improvement indices
- routes at each stage
- entering and exiting cells at each stage.

You must also explain how you know that your final solution is optimal. **(12 marks)**

## 1.5 Linear programming

The transportation problem requires you to minimise a certain value (the cost of the route) subject to certain constraints (the supply and demand at each location). You can formulate this as a linear programming problem.

**Links** You need to define your variables carefully when formulating a linear programming problem. ← **D1, Chapter 6**

Consider our first example:

| | **Depot *W*** | **Depot *X*** | **Depot *Y*** | **Depot *Z*** | **Stock** |
|---|---|---|---|---|---|
| **Supplier *A*** | 180 | 110 | 130 | 290 | 14 |
| **Supplier *B*** | 190 | 250 | 150 | 280 | 16 |
| **Supplier *C*** | 240 | 270 | 190 | 120 | 20 |
| **Demand** | 11 | 15 | 14 | 10 | 50 |

Let $x_{AW}$ be the number of units transported from $A$ to $W$, and $x_{BZ}$ be the number of units transported from $B$ to $Z$, and so on then you have the following solution.

**Notation** $x_{AW}$, $x_{BY}$, $x_{CZ}$ and so on are called the decision variables.

| | **Depot *W*** | **Depot *X*** | **Depot *Y*** | **Depot *Z*** | **Stock** |
|---|---|---|---|---|---|
| **Supplier *A*** | $x_{AW}$ | $x_{AX}$ | $x_{AY}$ | $x_{AZ}$ | 14 |
| **Supplier *B*** | $x_{BW}$ | $x_{BX}$ | $x_{BY}$ | $x_{BZ}$ | 16 |
| **Supplier *C*** | $x_{CW}$ | $x_{CX}$ | $x_{CY}$ | $x_{CZ}$ | 20 |
| **Demand** | 11 | 15 | 14 | 10 | 50 |

**Note** In this case a lot of these entries will be empty – there will be only $3 + 4 - 1 = 6$ non-empty cells, and all the rest will be blank. However, since you do not yet know which will be empty, you allow for any of them to be non-empty as you formulate the problem.

The objective is to minimise the total cost, which will be calculated by finding the sum of the products of the number of units transported along each route and the cost of using that route.

**A** In this case the objective is

Minimise $C = 180x_{AW} + 110x_{AX} + 130x_{AY} + 290x_{AZ}$
$+ 190x_{BW} + 250x_{BX} + 150x_{BY} + 280x_{BZ}$
$+ 240x_{CW} + 270x_{CX} + 190x_{CY} + 120x_{CZ}$

The top row of the table gives us the first constraint:

The total provided by a supplier cannot be more than the stock available.

$x_{AW} + x_{AX} + x_{AY} + x_{AZ} \leqslant 14$

Constraints for suppliers $B$ and $C$ are written in the same way using the next two rows of the table.

Demand constraints are written using the columns of the table.

The first column gives us

$x_{AW} + x_{BW} + x_{CW} \geqslant 11$

The total transported to a destination cannot be less than the given demand.

There are three more demand constraints written using the next three columns.

**Watch out** Take care to show the subscripts for each variable correctly.

- **The standard way of presenting a transportation problem as a linear programming problem is to:**
  - **first define your decision variables**
  - **next write down the objective function, and state that you want to minimise it**
  - **finally write down the constraints**

## Example 13

Formulate the transportation problem below as a linear programming problem. You must state your decision variables, objective and constraints.

| | ***R*** | ***S*** | ***T*** | **Supply** |
|---|---|---|---|---|
| ***A*** | 3 | 3 | 2 | 25 |
| ***B*** | 4 | 2 | 3 | 40 |
| ***C*** | 3 | 4 | 3 | 31 |
| **Demand** | 30 | 30 | 36 | |

Let $x_{ij}$ be the number of units transported from $i$ to $j$
where $i \in \{A, B, C\}$
$j \in \{R, S, T\}$
and $x_{ij} \geqslant 0$

Start by defining your decision variables.

Minimise $C = 3x_{AR} + 3x_{AS} + 2x_{AT}$
$+ 4x_{BR} + 2x_{BS} + 3x_{BT}$
$+ 3x_{CR} + 4x_{CS} + 3x_{CT}$

Next write down the objective function and state that your intention is to minimise it.

A

subject to: $x_{AR} + x_{AS} + x_{AT} \leqslant 25$
$x_{BR} + x_{BS} + x_{BT} \leqslant 40$
$x_{CR} + x_{CS} + x_{CT} \leqslant 31$
$x_{AR} + x_{BR} + x_{CR} \geqslant 30$
$x_{AS} + x_{BS} + x_{CS} \geqslant 30$
$x_{AT} + x_{BT} + x_{CT} \geqslant 36$

Write down the constraints. The supply constraints use $\leqslant$ but the demand constraints use $\geqslant$.

Make sure that the subscripts used for your decision variables are consistent with the values given for $i$ and $j$.

**Notation** You could also write the constraints using sigma notation.

For example, the first supply constraint would be

$$\sum x_{Aj} \leqslant 25$$

and the final demand constraint would be

$$\sum x_{iT} \geqslant 36$$

**Problem-solving**

For balanced transportation problems, the constraints may also be written as equalities. However, you should not mix these notations. Use either all equalities, or all (non-strict) inequalities.

So, for example, the first supply constraint may be written as

$$\sum x_{Aj} = 25$$

and the final demand constraint may be written as

$$\sum x_{iT} = 36$$

## Exercise 1E

**In questions 1–4**, formulate the transportation problems as linear programming problems.

**1**

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *A* | 150 | 213 | 222 | 32 |
| *B* | 175 | 204 | 218 | 44 |
| *C* | 188 | 198 | 246 | 34 |
| Demand | 28 | 45 | 37 | |

**2**

| | *P* | *Q* | *R* | *S* | Supply |
|---|---|---|---|---|---|
| *A* | 27 | 33 | 34 | 41 | 54 |
| *B* | 31 | 29 | 37 | 30 | 67 |
| *C* | 40 | 32 | 28 | 35 | 29 |
| Demand | 21 | 32 | 51 | 46 | |

**3**

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *A* | 17 | 24 | 19 | 123 |
| *B* | 15 | 21 | 25 | 143 |
| *C* | 19 | 22 | 18 | 84 |
| *D* | 20 | 27 | 16 | 150 |
| Demand | 200 | 100 | 200 | |

**4**

| | *P* | *Q* | *R* | *S* | Supply |
|---|---|---|---|---|---|
| *A* | 56 | 86 | 80 | 61 | 134 |
| *B* | 59 | 76 | 78 | 65 | 203 |
| *C* | 62 | 70 | 57 | 67 | 176 |
| *D* | 60 | 68 | 75 | 71 | 187 |
| Demand | 175 | 175 | 175 | 175 | |

A E/P **5** Three warehouses $A$, $B$ and $C$ supply four shops $W$, $X$, $Y$ and $Z$ with televisions. The cost in pounds of transporting a television from each warehouse to each shop is shown in the table. It also shows the number of televisions held at each warehouse and the number needed by each shop. The total transport cost is to be minimised.

| | $W$ | $X$ | $Y$ | $Z$ | Supply |
|---|---|---|---|---|---|
| $A$ | 8 | 11 | 7 | 9 | 25 |
| $B$ | 12 | 10 | 8 | 7 | 28 |
| $C$ | 10 | 12 | 9 | 8 | 21 |
| **Demand** | 20 | 15 | 12 | 16 | |

**Problem-solving**

Make sure you define decision variables for the dummy location. They won't be included in the objective function (because the associated costs are 0) but you will need to include constraints for them.

**a** Show that the problem is unbalanced. **(1 mark)**

**b** Formulate the problem as a linear programming problem. You must define your decision variables, your objective function and your constraints. You do not need to solve the problem. **(6 marks)**

E/P **6** The table shows the cost in pounds of transporting cars from three storage areas $A$, $B$ and $C$ to three showrooms $X$, $Y$ and $Z$. The table also shows the total number of cars stored and the total demand for cars at each showroom. The cost of transportation is to be minimised.

| | $X$ | $Y$ | $Z$ | Supply |
|---|---|---|---|---|
| $A$ | 70 | 50 | 60 | 12 |
| $B$ | 85 | 60 | 74 | 8 |
| $C$ | 68 | 73 | 80 | 10 |
| **Demand** | 11 | 9 | 6 | |

Jason attempts to formulate the problem as a linear programming problem as follows.

$C = 70x_{AX} + 50x_{AY} + 60x_{AZ} + 85x_{BX} + 60x_{BY} + 74x_{BZ} + 68x_{CX} + 73x_{CY} + 80x_{CZ}$

Subject to:

$x_{AX} + x_{AY} + x_{AZ} \leqslant 12$

$x_{BX} + x_{BY} + x_{BZ} \leqslant 8$

$x_{CX} + x_{CY} + x_{CZ} \leqslant 20$

$x_{AX} + x_{BX} + x_{CX} \leqslant 11$

$x_{AY} + x_{BY} + x_{CY} \leqslant 9$

$x_{AZ} + x_{BZ} + x_{CZ} \leqslant 6$

**a** Identify three errors in this formulation. **(3 marks)**

**b** Correctly formulate this problem as a linear programming problem. **(3 marks)**

## Mixed exercise 1

**1** The table shows the cost, in pounds, of transporting a car from each of three factories $A$, $B$ and $C$ to each of two showrooms $L$ and $M$. It also shows the number of cars available for delivery at each factory and the number required at each showroom.

| | $L$ | $M$ | Supply |
|---|---|---|---|
| $A$ | 20 | 70 | 15 |
| $B$ | 40 | 30 | 5 |
| $C$ | 60 | 90 | 8 |
| **Demand** | 16 | 12 | |

The total transportation cost is to be minimised.

**a** Use the north-west corner method to find an initial solution. **(1 mark)**

**b** Solve the transportation problem, stating shadow costs, improvement indices, entering cells, stepping-stone routes, $\theta$ values and exiting cells. **(4 marks)**

**c** Demonstrate that your solution is optimal and find the cost of your optimal solution. **(2 marks)**

**d** Formulate this problem as a linear programming problem, making your decision variables, objective function and constraints clear. **(5 marks)**

**2** The table shows the cost of transporting one unit of stock from each of three supply points $F$, $G$ and $H$ to each of three sales points $P$, $Q$ and $R$. It also shows the stock held at each supply point and the amount required at each sales point.

| | $P$ | $Q$ | $R$ | Supply |
|---|---|---|---|---|
| $F$ | 23 | 21 | 22 | 15 |
| $G$ | 21 | 23 | 24 | 35 |
| $H$ | 22 | 21 | 23 | 10 |
| **Demand** | 10 | 30 | 20 | |

**a** Use the north-west corner method to obtain an initial solution. **(2 marks)**

**b** Taking the most negative improvement index to indicate the entering square, perform two complete iterations of the stepping-stone method. You must state your shadow costs, improvement indices, entering cells, stepping-stone routes and exiting cells. **(4 marks)**

**c** Explain how you can tell that your current solution is optimal. **(1 mark)**

**d** State the cost of your optimal solution. **(1 mark)**

**e** Taking the zero improvement index to indicate the entering cell, perform one further iteration to obtain a second optimal solution. **(2 marks)**

A E/P **3** The table below shows the costs in pounds of transporting one unit of stock from each of four warehouses $J$, $K$, $L$ and $M$ to each of three retailers $X$, $Y$ and $Z$. The available stock at each warehouse and the demand at each retailer is also shown. The total cost is to be minimised.

| | $X$ | $Y$ | $Z$ | **Supply** |
|---|---|---|---|---|
| $J$ | 8 | 5 | 7 | 30 |
| $K$ | 5 | 5 | 9 | 40 |
| $L$ | 7 | 2 | 10 | 50 |
| $M$ | 6 | 3 | 15 | 50 |
| **Demand** | 25 | 45 | 100 | |

A possible north-west corner solution is

| | $X$ | $Y$ | $Z$ | **Supply** |
|---|---|---|---|---|
| $J$ | 25 | 5 | | 30 |
| $K$ | | 40 | | 40 |
| $L$ | | 0 | 50 | 50 |
| $M$ | | | 50 | 50 |
| **Demand** | 25 | 45 | 100 | |

**a** Explain why it is was necessary to add a zero entry (in cell $LY$) to the solution. **(1 mark)**

**b** State the cost of this initial solution. **(1 mark)**

**c** Choosing cell $MX$ as the entering cell, perform one iteration of the stepping-stone method to obtain an improved solution. You must make your routes clear, state your exiting cell and the cost of the improved solution. **(3 marks)**

**d** Determine whether your current solution is optimal. Give a reason for your answer. **(1 mark)**

After two more iterations the following solution was found.

| | $X$ | $Y$ | $Z$ | **Supply** |
|---|---|---|---|---|
| $J$ | | | 30 | 30 |
| $K$ | | 20 | 20 | 40 |
| $L$ | | | 50 | 50 |
| $M$ | 25 | 25 | | 50 |
| **Demand** | 25 | 45 | 100 | |

**e** Taking the most negative improvement index to indicate the entering square, perform one further complete iteration of the stepping-stone method to obtain an optimal solution. You must state your shadow costs, improvement indices, stepping-stone route and exiting cell. **(4 marks)**

**f** Due to increased bridge tolls, the cost of supplying each unit from warehouse $J$ to retailer $Y$ is doubled. State, with reasons, whether this will affect the total cost of the optimal solution. **(1 mark)**

A E/P **4** **a** Explain why a dummy demand point might be needed when solving a transportation problem. **(1 mark)**

The table shows the cost, in pounds, of transporting one van load of fruit-tree seedlings from each of three greenhouses *A*, *B* and *C* to three garden centres *S*, *T* and *U*. It also shows the stock held at each greenhouse and the amount required at each garden centre.

| | *S* | *T* | *U* | Supply |
|---|---|---|---|---|
| *A* | 6 | 10 | 7 | 50 |
| *B* | 7 | 5 | 8 | 70 |
| *C* | 6 | 7 | 7 | 50 |
| **Demand** | 100 | 30 | 20 | |

The total cost of transportation is to be minimised.

**b** Use the north-west corner method to obtain an initial solution. **(2 marks)**

**c** Taking the most negative improvement index in each case to indicate the entering square, use the stepping-stone method to obtain an optimal solution. You must state your shadow costs, improvement indices, stepping-stone routes, entering squares and exiting cells. **(4 marks)**

**d** State the cost of your optimal solution. **(1 mark)**

**e** Formulate this transportation problem as a linear programming problem. You must define your decision variables and make the constraints and objective function clear.

You do not need to solve the problem. **(5 marks)**

E **5** Three manufacturers *A*, *B* and *C* supply rolls of carpet to four stores *P*, *Q*, *R* and *S*. The table gives the cost in pounds of transporting a roll of carpet from each manufacturer to each store. It also shows the number of rolls of carpet held at each manufacturer and the number of rolls of carpet required by each store. The total cost of transportation is to be minimised.

| | *P* | *Q* | *R* | *S* | Supply |
|---|---|---|---|---|---|
| *A* | 28 | 12 | 19 | 16 | 28 |
| *B* | 31 | 28 | 23 | 19 | 33 |
| *C* | 18 | 21 | 22 | 28 | 18 |
| **Demand** | 16 | 20 | 26 | 17 | |

Formulate this transportation problem as a linear programming problem. You must define your decision variables and make the constraints and objective function clear.

You do not need to solve the problem. **(6 marks)**

## Challenge

**A** The table shows the transportation costs, in pounds, for each unit of stock from three warehouses $A$, $B$ and $C$ to three retailers $P$, $Q$ and $R$. It also shows the availability of stock at the warehouses, and the demand at each of the retailers, over a period of one week.

| | $P$ | $Q$ | $R$ | **Stock** |
|---|---|---|---|---|
| $A$ | 7 | 8 | 6 | 14 |
| $B$ | 5 | 7 | 9 | 12 |
| $C$ | 6 | 8 | 8 | 16 |
| **Demand** | 15 | 9 | 11 | |

Any stock that remains in a warehouse at the end of the week incurs a storage charge. The weekly storage charges for each unit of surplus stock are £3 at warehouse $A$, £5 at warehouse $B$ and £4 at warehouse $C$. The total weekly cost, including storage charges, is to be minimised.

**a** Add a suitably labelled dummy to the table to represent the problem.

**b** Formulate the problem as a linear programming problem.

## Summary of key points

**1** You need to be familiar with the following terminology:

- The capacity of each of the **supply points** (or **sources**) – the quantity of goods that can be produced at each factory or held at each warehouse. This is called the **supply** or **stock**.
- The amount required at each of the **demand points** (or **destinations**) – the quantity of goods that are needed at each shop or by each customer. This is called the **demand**.
- The **unit cost** of transporting goods from the supply points to the demand points.

**2** The steps in the **transportation algorithm** are as follows:

- First find an initial solution that uses all the stock and meets all the demands.
- Calculate the total cost of this solution and see if it can be reduced by transporting some goods along a route not currently in the solution. (If this is not possible then the solution is optimal.)
- If the cost can be reduced by using a new route, allocate as many units as possible to this new route to create a new solution.
- Check the new solution in the same way as the initial solution to see if it is optimal. If not, repeat the previous step.
- When no further savings are possible, an optimal solution has been found.

**3** The **north-west corner method**:

- Create a table, with one row for every source and one column for every destination. Each cell represents a route from a source to a destination. Each destination's demand is given at the foot of each column and each source's stock is given at the end of each row. Enter numbers in each cell to show how many units are to be sent along that route.

A

- Begin with the top left-hand corner (the north-west corner). Allocate the maximum available quantity to meet the demand at this destination (but do not exceed the stock at this source).
- As each stock is emptied, move one square down and allocate as many units as possible from the next source until the demand of the destination is met. As each demand is met, move one square to the right and again allocate as many units as possible.
- Stop when all the stock is assigned and all the demands are met.

**4** When total supply $\neq$ total demand, a transportation problem is unbalanced.

- If total supply $>$ total demand, you need to add a dummy demand point.
- If total supply $<$ total demand, you need to add a dummy supply point.

In each case the demand or supply at the dummy is chosen so that total demand is equal to total supply, and the transportation costs to/from the dummy location are all zero.

**5**
- In a feasible solution to a transportation problem with $m$ rows and $n$ columns, if the number of cells used is less than $n + m - 1$, then the solution is **degenerate**.
- This will happen when an entry, other than the last, is made that satisfies the supply for a given row, and at the same time satisfies the demand for a given column.
- The algorithm requires that $n + m - 1$ cells are used in every solution, so a zero needs to be placed in a currently unused cell.

**6** To find an improved solution, you need to:

- Use the non-empty cells to find the **shadow costs**.
- Use the shadow costs and the empty cells to find **improvement indices**.
- Use the improvement indices and the **stepping-stone method** to find an improved solution.

**7** Transportation costs are made up of two components, one associated with the source and one with the destination. These costs of using that route are called **shadow costs**.

**8** To find the **shadow costs**, follow these steps:

- Start with the north-west corner, and set the cost linked with its source to zero.
- Move along the row to any other non-empty squares. Set the cost linked with these destinations equal to the total transportation cost for that route (since the source cost for the first row is 0).
- When all possible destination costs for that row have been established, go to the start of the next row.
- Move along this row to any non-empty squares and use the destination costs found earlier, to establish the source cost for the row. Once that has been done, find any further unknown destination costs.
- Repeat the previous two steps until all source and destination costs have been found.

A

**9** • The improvement index in sending a unit from a source $P$ to a demand point $Q$ is found by subtracting the source cost $S(P)$ and destination cost $D(Q)$ from the stated cost of transporting one unit along that route $C(PQ)$:

Improvement index for $PQ = I_{PQ} = C(PQ) - S(P) - D(Q)$

- The route with the most negative improvement index will be introduced into the solution.
- The cell corresponding to the value with the most negative improvement index becomes the **entering cell** (or **entering square** or **entering route**) and the route it replaces is referred to as the **exiting cell** (or **exiting square** or **exiting route**).
- If there are two equal potential entering cells you may choose either. Similarly, if there are two equal exiting cells, you may select either.
- If there are no negative improvement indices the solution is **optimal**.

**10 The stepping-stone method**

- Create the cycle of adjustments. The two basic rules are:
  - Within any row and any column there can only be one increasing cell and one decreasing cell.
  - Apart from the entering cell, adjustments are only made to non-empty cells.
- Once the cycle of adjustments has been found, you transfer the maximum number of units through this cycle. This will be equal to the smallest number in the decreasing cells (since you may not have negative units being transported).
- You then adjust the solution to incorporate this improvement.

**11** • One entry that has been reduced to 0 by the stepping-stone method is called the **exiting cell**, and should be left blank in the improved solution.

- At each iteration you create one entering cell and one exiting cell.
- To find an optimal solution, continue to calculate new shadow costs and improvement indices and then apply the stepping-stone method. Repeat this iteration until all the improvement indices are non-negative.

**12** The standard way of presenting a transportation problem as a linear programming problem is to:

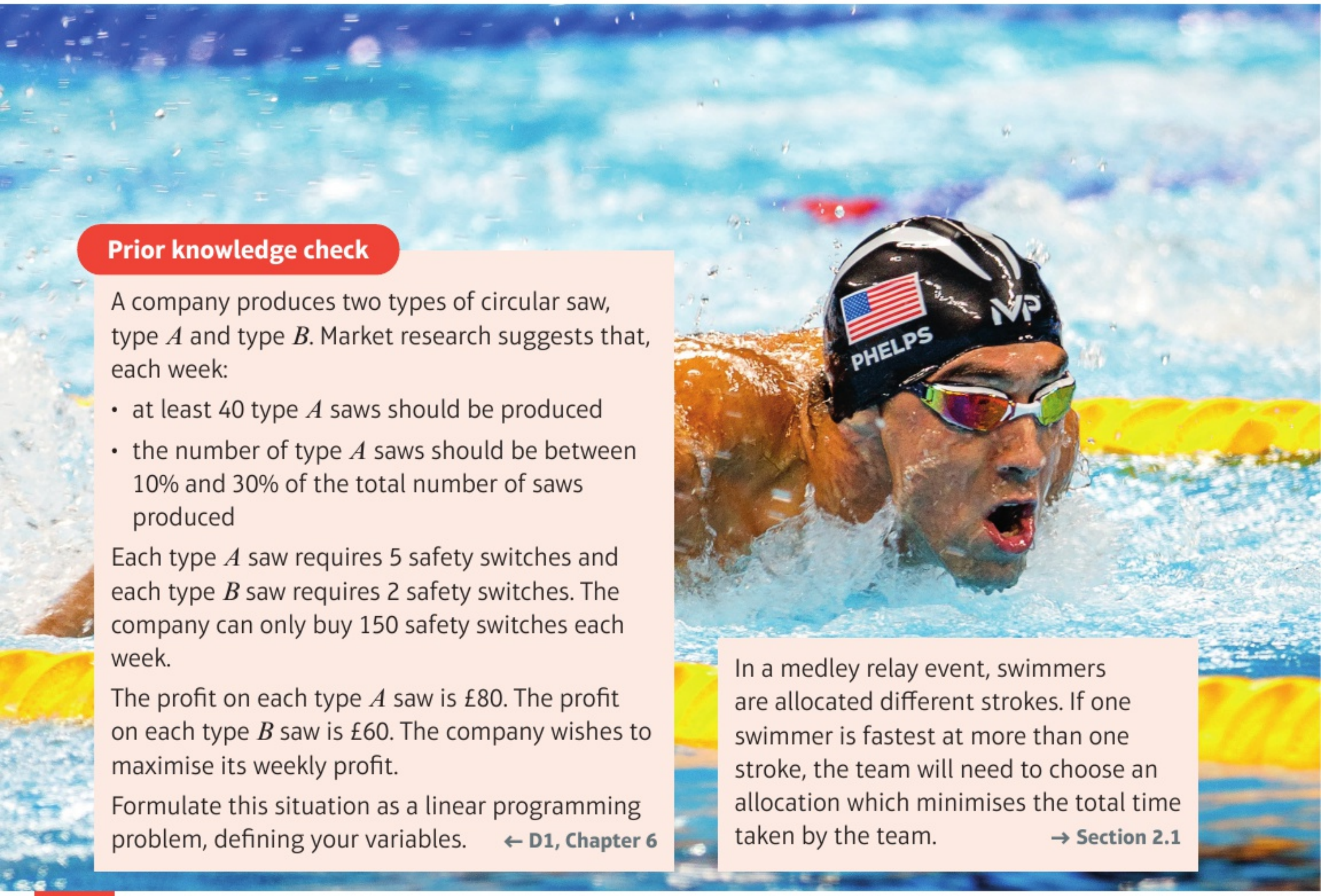
- first define your decision variables
- next write down the objective function, and state that you want to minimise it
- finally write down the constraints.

# 2 Allocation problems

**Objectives**

After completing this chapter you should be able to:

- Reduce cost matrices → pages 39–71
- Use the Hungarian algorithm to find a least cost allocation → pages 39–49
- Adapt the Hungarian algorithm to use a dummy → pages 49–51
- Modify the Hungarian algorithm to deal with a maximum profit allocation → pages 51–55
- Adapt the Hungarian algorithm to manage incomplete data → pages 56–62
- Formulate allocation problems as linear programming problems → pages 62–66




**Prior knowledge check**

A company produces two types of circular saw, type $A$ and type $B$. Market research suggests that, each week:

- at least 40 type $A$ saws should be produced
- the number of type $A$ saws should be between 10% and 30% of the total number of saws produced

Each type $A$ saw requires 5 safety switches and each type $B$ saw requires 2 safety switches. The company can only buy 150 safety switches each week.

The profit on each type $A$ saw is £80. The profit on each type $B$ saw is £60. The company wishes to maximise its weekly profit.

Formulate this situation as a linear programming problem, defining your variables. ← D1, Chapter 6

In a medley relay event, swimmers are allocated different strokes. If one swimmer is fastest at more than one stroke, the team will need to choose an allocation which minimises the total time taken by the team. → Section 2.1

## 2.1 The Hungarian algorithm

In a swimming relay race there are four team members and each one swims just one of the strokes: butterfly, backstroke, breaststroke or front-crawl.

It is the total time taken that determines how well a team has done and so each team member must be allocated the stroke that will minimise the total time.

This is typical of allocation problems in which workers must be assigned to tasks to minimise costs.

In allocation problems each worker must do just one task and each task must be done by just one worker. We are seeking a one-to-one solution.

- **This means that we require the same number of tasks as workers.**

**Note** The 'cost' might be time, money, or some other quantity.

In finding a solution to an allocation problem, only **relative** costs are important.

In a typical problem, information about costs is presented in a table, called a **cost matrix**.

Here is the cost matrix for a swimming relay team. The costs in the matrix are times in seconds.

| | Butterfly | Backstroke | Breaststroke | Front-crawl |
|---|---|---|---|---|
| **Adam** | 35 | 36 | 36 | 25 |
| **Boris** | 31 | 31 | 35 | 23 |
| **Carl** | 33 | 35 | 39 | 26 |
| **Dean** | 32 | 38 | 33 | 24 |

Adam's fastest time is 25 seconds for the front-crawl. To compare his times across the strokes, you can subtract 25 from each value in the top row to give his relative times.

The same can be done for the other swimmers, in each case subtracting the smallest value in a row from every value in the row.

The cost matrix now looks like this.

| | Butterfly | Backstroke | Breaststroke | Front-crawl |
|---|---|---|---|---|
| **Adam** | 10 | 11 | 11 | 0 |
| **Boris** | 8 | 8 | 12 | 0 |
| **Carl** | 7 | 9 | 13 | 0 |
| **Dean** | 8 | 14 | 9 | 0 |

This shows that front-crawl is the fastest stroke for each swimmer. The quickest way to complete the race would be for every swimmer to swim front-crawl, but this is not permitted. The other entries in the table show how much **additional** time would be needed for each possible allocation of swimmers to the other three strokes.

The same process is now carried out with the columns, to give:

| | Butterfly | Backstroke | Breaststroke | Front-crawl |
|---|---|---|---|---|
| **Adam** | 3 | 3 | 2 | 0 |
| **Boris** | 1 | 0 | 3 | 0 |
| **Carl** | 0 | 1 | 4 | 0 |
| **Dean** | 1 | 6 | 0 | 0 |

One swimmer must be allocated to breaststroke. This entry shows that allocating breaststroke to Dean minimises the total **additional** time incurred by the team. Similarly, the total team time is minimised by allocating butterfly to Carl, and backstroke to Boris. Adam is not reallocated, and will swim front-crawl.

This matrix is called the **reduced cost matrix** and finding it is one of the key stages in solving allocation problems.

- **To reduce the cost matrix:**
  - **subtract the least value in each row from each element of that row**
  - **using the new matrix, subtract the least value in each column from each element in that column**

## Example 1

The table shows the times taken, in minutes, by four workers to complete each task of a four-stage production process.

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| **Kris** | 12 | 23 | 15 | 40 |
| **Laura** | 14 | 21 | 17 | 20 |
| **Sam** | 13 | 22 | 20 | 30 |
| **Steve** | 14 | 24 | 13 | 10 |

The time taken for the process is the sum of the times of the four separate tasks. Reduce the cost matrix.

The smallest numbers in rows 1, 2, 3 and 4 are 12, 14, 13 and 10 respectively. If we subtract these numbers from each element in the row our table becomes

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| Kris | 0 | 11 | 3 | 28 |
| Laura | 0 | 7 | 3 | 6 |
| Sam | 0 | 9 | 7 | 17 |
| Steve | 4 | 14 | 3 | 0 |

After you have reduced the rows, each row will contain at least one zero entry.

The smallest numbers in columns 1, 2, 3 and 4 are 0, 7, 3 and 0 respectively. We now subtract these numbers from each element in the column.

**Watch out** When you are reducing a cost matrix you should reduce the **rows first**.

The table becomes

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| Kris | 0 | 4 | 0 | 28 |
| Laura | 0 | 0 | 0 | 6 |
| Sam | 0 | 2 | 4 | 17 |
| Steve | 4 | 7 | 0 | 0 |

This is the reduced cost matrix.

You can use the reduced cost matrix to find an **optimal allocation** of tasks to workers.

- **If it is possible to allocate tasks to workers in such a way that the corresponding entries in the reduced cost matrix are all zero, then this allocation is optimal.**

In the above worked example, an optimal allocation is

Kris $\to Y$

Laura $\to X$

Sam $\to W$

Steve $\to Z$

**Watch out** An optimal allocation, here, is one that **minimises** the 'cost' (in this case, the total time taken). There may be more than one optimal allocation but all will have the same cost.

To find the total cost of this allocation you can look at the original table:

$15 + 21 + 13 + 10 = 59$

This is also the total value of the row and column reductions made:

$12 + 14 + 13 + 10 + 7 + 3 = 59$

You can use the Hungarian algorithm to find a least cost allocation.

### The Hungarian algorithm

**1** Start by finding the reduced cost matrix. Reduce rows first.

**2** Determine the minimum number of straight lines (horizontal or vertical), which will cover all of the zeros in the matrix.

**3** In an $n \times n$ matrix, if you cannot cover them in fewer than $n$ lines, you have an optimal solution and you stop.

**4** If in an $n \times n$ matrix you can cover the zeros with fewer than $n$ lines, drawn vertically or horizontally, the solution can be improved.

**5** Draw in these lines and look for the smallest uncovered element, $e$.

**6** Add $e$ to the elements in each covered row and each covered column, adding it twice to any element covered twice.

**7** Subtract $e$ from every element of the matrix.

**8** Repeat steps **2** to **7** until an optimal solution is found.

**Notation** The process in steps **5** to **7** is sometimes called **augmenting** the reduced cost matrix by $e$.

## Example 2

Barbara, Percival and Spike do garden maintenance. The table shows the times, in minutes, that they would take to do each task in Mrs Green's garden. Allocate the tasks so that the time taken to do the whole job is as small as possible.

| | Dig vegetable patch | Weed flower beds | Cut lawn and hedges |
|---|---|---|---|
| **Barbara** | 250 | 80 | 160 |
| **Percival** | 230 | 90 | 150 |
| **Spike** | 230 | 110 | 140 |

| | Dig vegetable patch | Weed flower beds | Cut lawn and hedges |
|---|---|---|---|
| Barbara | 170 | 0 | 80 |
| Percival | 140 | 0 | 60 |
| Spike | 120 | 0 | 30 |

First, reduce the cost matrix.

Subtract 80, 90 and 110 from rows 1, 2 and 3 respectively.

| | Dig vegetable patch | Weed flower beds | Cut lawn and hedges |
|---|---|---|---|
| Barbara | 50 | 0 | 50 |
| Percival | 20 | 0 | 30 |
| Spike | 0 | 0 | 0 |

Subtract 120, 0 and 30 from columns 1, 2 and 3 respectively.

We can cover all the zeros using only two lines like this:

| | Dig vegetable patch | Weed flower beds | Cut lawn and hedges |
|---|---|---|---|
| Barbara | 50 | 0 | 50 |
| Percival | 20 | 0 | 30 |
| Spike | 0 | 0 | 0 |

Next, test for optimality. Since this is a 3 × 3 matrix, you will have an optimal solution if you **have** to use three lines (drawn vertically or horizontally) to cover all the zeros.

If you can cover them in fewer than three lines, the solution is not optimal.

So we have not yet found an optimal (minimum cost) solution – there is no way of allocating each worker to a task using just the zero cost cells.

The minimum uncovered element is 20.

We add 20 to each row covered by a line:

| | Dig vegetable patch | Weed flower beds | Cut lawn and hedges |
|---|---|---|---|
| Barbara | 50 | 0 | 50 |
| Percival | 20 | 0 | 30 |
| Spike | 20 | 20 | 20 |

and then add 20 to to each column covered by a line:

| | Dig vegetable patch | Weed flower beds | Cut lawn and hedges |
|---|---|---|---|
| Barbara | 50 | 20 | 50 |
| Percival | 20 | 20 | 30 |
| Spike | 20 | 40 | 20 |

This entry was covered by two lines, so 20 was added to it twice.

We then subtract 20 from each element in the matrix:

| | Dig vegetable patch | Weed flower beds | Cut lawn and hedges |
|---|---|---|---|
| Barbara | 30 | 0 | 30 |
| Percival | 0 | 0 | 10 |
| Spike | 0 | 20 | 0 |

We have to use three lines to cover all the zeros, so this solution is optimal.

A minimal allocation is

Barbara – weed flower beds
Percival – dig vegetable patch
Spike – cut lawn and hedges

Using the original cost matrix,
time taken = 80 + 230 + 140 = 450 minutes

You can check your total cost by considering the row and column reductions, plus any costs added to the matrix at each iteration of the algorithm.

If you add together the row and column reductions you get

80 + 90 + 110 + 120 + 0 + 30 = 430

If you now add on the extra 20 used in the single iteration you get

430 + 20 = 450

## A short cut

You can save time in your exam by using just one step to augment the reduced cost matrix.

If the smallest uncovered element is $e$:

- each element covered by two lines will increase by $e$
- each element covered by just one line will be unchanged
- each uncovered element will be reduced by $e$

## Example 3

Five workers Ben, Ellie, Greg, Hyo and Toby are to be assigned to five tasks involved in making a soft toy. The table shows the times in minutes taken to complete each task.

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| **Ben** | 9 | 8 | 3 | 6 | 10 |
| **Ellie** | 5 | 5 | 7 | 5 | 5 |
| **Greg** | 10 | 9 | 3 | 9 | 10 |
| **Hyo** | 10 | 7 | 2 | 9 | 7 |
| **Toby** | 9 | 8 | 2 | 7 | 10 |

Reducing rows first, use the Hungarian algorithm to determine an allocation that minimises the total time.

Reducing rows first:

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| Ben | 6 | 5 | 0 | 3 | 7 |
| Ellie | 0 | 0 | 2 | 0 | 0 |
| Greg | 7 | 6 | 0 | 6 | 7 |
| Hyo | 8 | 5 | 0 | 7 | 5 |
| Toby | 7 | 6 | 0 | 5 | 8 |

We cannot reduce the columns, since there is a zero already in each column.

We can cover all the zeros in just two lines like this:

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| Ben | 6 | 5 | 0 | 3 | 7 |
| Ellie | 0 | 0 | 2 | 0 | 0 |
| Greg | 7 | 6 | 0 | 6 | 7 |
| Hyo | 8 | 5 | 0 | 7 | 5 |
| Toby | 7 | 6 | 0 | 5 | 8 |

**Method 1: Using the short cut**

The smallest uncovered element is 3, so using the short cut, we

- add 3 to the element covered by two lines
- leave the elements covered by just one line unchanged
- subtract 3 from the uncovered elements

This gives the following matrix:

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| Ben | 3 | 2 | 0 | 0 | 4 |
| Ellie | 0 | 0 | 5 | 0 | 0 |
| Greg | 4 | 3 | 0 | 3 | 4 |
| Hyo | 5 | 2 | 0 | 4 | 2 |
| Toby | 4 | 3 | 0 | 2 | 5 |

The shaded elements were all uncovered, so they have each had 3 subtracted from them.
The circled element was covered by two lines, so it has had 3 added to it.

**Method II: Not using the short cut**

- Add 3 to each element in the 'Ellie' row.

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| Ben | 6 | 5 | 0 | 3 | 7 |
| Ellie | 3 | 3 | 5 | 3 | 3 |
| Greg | 7 | 6 | 0 | 6 | 7 |
| Hyo | 8 | 5 | 0 | 7 | 5 |
| Toby | 7 | 6 | 0 | 5 | 8 |

- Add 3 to each element in the 'Fill' column.

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| Ben | 6 | 5 | 3 | 3 | 7 |
| Ellie | 3 | 3 | 8 | 3 | 3 |
| Greg | 7 | 6 | 3 | 6 | 7 |
| Hyo | 8 | 5 | 3 | 7 | 5 |
| Toby | 7 | 6 | 3 | 5 | 8 |

Notice that the element in the 'Ellie' row and the 'Fill' column has been increased by 6.

- Subtract 3 from all elements in the table.

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| Ben | 3 | 2 | 0 | 0 | 4 |
| Ellie | 0 | 0 | 5 | 0 | 0 |
| Greg | 4 | 3 | 0 | 3 | 4 |
| Hyo | 5 | 2 | 0 | 4 | 2 |
| Toby | 4 | 3 | 0 | 2 | 5 |

The matrix resulting from both methods can be covered in just three lines like this:

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| Ben | 3 | 2 | 0 | 0 | 4 |
| Ellie | 0 | 0 | 5 | 0 | 0 |
| Greg | 4 | 3 | 0 | 3 | 4 |
| Hyo | 5 | 2 | 0 | 4 | 2 |
| Toby | 4 | 3 | 0 | 2 | 5 |

**Problem-solving**

You can choose any arrangement of lines that covers all the zeros with the minimum possible number of lines. However, choosing the arrangement that makes the **minimum uncovered value** as large as possible can save you time. See 'A useful tip' following Example 5 on page 53.

**Use the short cut again.**

The minimum uncovered element is 2, so we

- add 2 to the elements covered by two lines
- leave the elements covered by just one line unchanged
- subtract 2 from the uncovered elements

This gives:

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| Ben | 1 | 0 | 0 | 0 | 2 |
| Ellie | 0 | 0 | (7) | (2) | 0 |
| Greg | 2 | 1 | 0 | 3 | 2 |
| Hyo | 3 | 0 | 0 | 4 | 0 |
| Toby | 2 | 1 | 0 | 2 | 3 |

The shaded elements were uncovered, so they have each had 2 subtracted from them.
The circled elements were covered by two lines, so they have each had 2 added to them.

The zeros in this table can be covered in four lines, so we have not yet found an optimal solution.

The four lines can be placed like this:

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| Ben | 1 | 0 | 0 | 0 | 2 |
| Ellie | 0 | 0 | 7 | 2 | 0 |
| Greg | 2 | 1 | 0 | 3 | 2 |
| Hyo | 3 | 0 | 0 | 4 | 0 |
| Toby | 2 | 1 | 0 | 2 | 3 |

**Use the short cut again.**

The minimum uncovered element is 1, so we

- add 1 to the element covered by two lines
- leave the elements covered by just one line unchanged
- subtract 1 from the uncovered elements

This gives:

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| Ben | 1 | 0 | (1) | 0 | 2 |
| Ellie | 0 | 0 | (8) | 2 | 0 |
| Greg | 1 | 0 | 0 | 2 | 1 |
| Hyo | 3 | 0 | (1) | 4 | 0 |
| Toby | 1 | 0 | 0 | 1 | 2 |

The shaded elements were uncovered, so they have each had 1 subtracted from them.
The circled elements were covered by two lines, so they have each had 1 added to them.

We need five lines to cover all the zeros in this table, so we have found our optimal solution.

In fact there are two solutions:

Either

Ben – Finish, Ellie – Cut, Greg – Fill,
Hyo – Pack, Toby – Sew

Or

Ben – Finish, Ellie – Cut, Greg – Sew,
Hyo – Pack, Toby – Fill

Both solutions have a total time of 29 minutes.
(6 + 5 + 3 + 7 + 8 or 6 + 5 + 9 + 7 + 2)

To find the solution from the final table in the worked example above, use logic to choose an optimum allocation.

First locate any single zeros in rows or columns, since these must be used.

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| **Ben** | 1 | 0 | 1 | 0 | 2 |
| **Ellie** | 0 | 0 | 8 | 2 | 0 |
| **Greg** | 1 | 0 | 0 | 2 | 1 |
| **Hyo** | 3 | 0 | 1 | 4 | 0 |
| **Toby** | 1 | 0 | 0 | 1 | 2 |

So we must include Ellie – Cut and Ben – Finish in our solution.

This leaves Greg, Hyo and Toby, and Sew, Fill and Pack to be assigned.

The only person now able to do Pack is Hyo, so we assign her.

| | Cut | Sew | Fill | Finish | Pack |
|---|---|---|---|---|---|
| **Ben** | 1 | 0 | 1 | 0 | 2 |
| **Ellie** | 0 | 0 | 8 | 2 | 0 |
| **Greg** | 1 | 0 | 0 | 2 | 1 |
| **Hyo** | 3 | 0 | 1 | 4 | 0 |
| **Toby** | 1 | 0 | 0 | 1 | 2 |

We now see that Greg and Toby must cover Sew and Fill, which both are able to do, giving our two solutions.

## Exercise 2A

**In questions 1 to 4**, the tables show the cost, in pounds, of allocating workers to tasks. Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, showing the table at each stage and state your final solution and its cost.

**1**

| | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|
| Worker $A$ | 34 | 35 | 31 |
| Worker $B$ | 26 | 31 | 27 |
| Worker $C$ | 30 | 37 | 32 |

**2**

| | Task $A$ | Task $B$ | Task $C$ | Task $D$ |
|---|---|---|---|---|
| Worker $P$ | 34 | 37 | 32 | 32 |
| Worker $Q$ | 35 | 32 | 34 | 37 |
| Worker $R$ | 42 | 35 | 37 | 36 |
| Worker $S$ | 38 | 34 | 35 | 39 |

**3**

| | Task $R$ | Task $S$ | Task $T$ | Task $U$ |
|---|---|---|---|---|
| Worker $J$ | 20 | 22 | 14 | 24 |
| Worker $K$ | 20 | 19 | 12 | 20 |
| Worker $L$ | 13 | 10 | 18 | 16 |
| Worker $M$ | 22 | 23 | 9 | 28 |

**4**

| | Task $V$ | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|---|
| Worker $D$ | 85 | 95 | 97 | 87 | 80 |
| Worker $E$ | 110 | 115 | 95 | 105 | 100 |
| Worker $F$ | 90 | 95 | 86 | 93 | 105 |
| Worker $G$ | 85 | 83 | 84 | 85 | 87 |
| Worker $H$ | 100 | 100 | 105 | 120 | 95 |

E/P **5** A junior school has to enter four pupils in an athletics competition comprising four events; 100 m, hurdles, 200 m, 400 m. The rules are that each pupil may only enter one event and the winning team is the one whose total time for the four events is the least. The school holds trials and the table shows the time, in seconds, that each of the team members takes.

| | 100 m | Hurdles | 200 m | 400 m |
|---|---|---|---|---|
| Ahmed | 14 | 21 | 37 | 64 |
| Ben | 13 | 22 | 40 | 68 |
| Chang | 12 | 20 | 38 | 70 |
| Davina | 13 | 21 | 39 | 74 |

**a** Reducing rows first, obtain the reduced cost matrix. **(3 marks)**

**b** Use the Hungarian algorithm to determine which event each pupil should be allocated to. **(6 marks)**

**c** Explain how you know your solution is optimal. **(1 mark)**

(E) **6** The table shows the cost, in pounds, of purchasing six types of tree from six local nurseries.

| | Beech | Elm | Eucalyptus | Oak | Olive | Willow |
|---|---|---|---|---|---|---|
| **Nursery *A*** | 153 | 87 | 62 | 144 | 76 | 68 |
| **Nursery *B*** | 162 | 105 | 87 | 152 | 88 | 72 |
| **Nursery *C*** | 159 | 84 | 75 | 165 | 79 | 77 |
| **Nursery *D*** | 145 | 98 | 63 | 170 | 85 | 81 |
| **Nursery *E*** | 149 | 94 | 70 | 138 | 82 | 89 |
| **Nursery *F*** | 160 | 92 | 82 | 147 | 80 | 85 |

A landscape gardener wishes to support each of these local nurseries for the year and so decides to use each nursery to supply one type of tree. He will use equal numbers of each type of tree throughout the year.

**a** Reducing rows first, use the Hungarian algorithm to determine which type of tree should be supplied by which nursery to minimise the total cost. **(8 marks)**

**b** Find the minimum cost. **(2 marks)**

## 2.2 Using a dummy

You can adapt the Hungarian algorithm to make the number of rows match the number of columns.

- **If you have to solve an allocation problem which is not $n \times n$, you can introduce dummy rows or dummy columns with zero entries.**

**Links** This is similar to the way in which you handle a transportation problem which is not $n \times n$. ← Section 1.2

In this situation you will end up with either:

- workers allocated to dummy tasks – these workers are not allocated to any task

or:

- tasks allocated to dummy workers – these tasks will not be completed

## Example 4

The table shows the time, in minutes, taken for four workers Mark, Nicky, Nigel and Susie to do each of three tasks $A$, $B$ and $C$.

| | Task $A$ | Task $B$ | Task $C$ |
|---|---|---|---|
| **Mark** | 12 | 23 | 15 |
| **Nicky** | 14 | 21 | 17 |
| **Nigel** | 13 | 22 | 20 |
| **Susie** | 14 | 24 | 13 |

Use the Hungarian algorithm to obtain an allocation that minimises the total time.

| | Task $A$ | Task $B$ | Task $C$ | Task $D$ |
|---|---|---|---|---|
| Mark | 12 | 23 | 15 | 0 |
| Nicky | 14 | 21 | 17 | 0 |
| Nigel | 13 | 22 | 20 | 0 |
| Susie | 14 | 24 | 13 | 0 |

Continue as usual.

| | Task $A$ | Task $B$ | Task $C$ | Task $D$ |
|---|---|---|---|---|
| Mark | 0 | 2 | 2 | 0 |
| Nicky | 2 | 0 | 4 | 0 |
| Nigel | 1 | 1 | 7 | 0 |
| Susie | 2 | 3 | 0 | 0 |

It takes four straight lines to cover the zeros, so we have an optimal solution:

Mark – task $A$
Nicky – task $B$
Nigel – task $D$ (dummy)
Susie – task $C$

The total time taken is
$12 + 21 + 0 + 13 = 46$ minutes

**Problem-solving**

There are more workers than tasks, so introduce a dummy column so that the number of rows is equal to the number of columns.

Fill the extra column with zeros, since the dummy task $D$ will take no time to complete.

There are no row subtractions to do (each row contains a zero), so subtract 12, 21, 13 and 0 from the columns.

## Exercise 2B

**In questions 1 to 4**, the tables show the cost, in pounds, of allocating workers to tasks. Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, show the table at each stage, and state your final solution and its cost.

**1**

| | Task $M$ | Task $N$ |
|---|---|---|
| **Worker $J$** | 23 | 26 |
| **Worker $K$** | 26 | 30 |
| **Worker $L$** | 29 | 28 |

2

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| Worker $A$ | 31 | 43 | 19 | 35 |
| Worker $B$ | 28 | 46 | 10 | 34 |
| Worker $C$ | 24 | 42 | 13 | 33 |

3

| | Task $R$ | Task $S$ | Task $T$ |
|---|---|---|---|
| Worker $W$ | 81 | 45 | 55 |
| Worker $X$ | 67 | 32 | 48 |
| Worker $Y$ | 87 | 38 | 58 |
| Worker $Z$ | 73 | 37 | 60 |

4

| | Task $E$ | Task $F$ | Task $G$ | Task $H$ |
|---|---|---|---|---|
| Worker $P$ | 24 | 42 | 32 | 31 |
| Worker $Q$ | 22 | 39 | 30 | 35 |
| Worker $R$ | 13 | 34 | 22 | 25 |
| Worker $S$ | 19 | 41 | 27 | 29 |
| Worker $T$ | 18 | 40 | 31 | 33 |

E/P **5** Five workers $A$, $B$, $C$, $D$ and $E$ are to be assigned to six tasks 1, 2, 3, 4, 5 and 6. Each worker can only be assigned to one task and each assigned task is to be completed by one worker.

The table shows the cost, in pounds, of assigning each worker to each task.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $A$ | 53 | 75 | 58 | 60 | 75 | 78 |
| $B$ | 44 | 81 | 64 | 55 | 78 | 77 |
| $C$ | 51 | 72 | 51 | 61 | 81 | 72 |
| $D$ | 60 | 77 | 60 | 55 | 76 | 73 |
| $E$ | 51 | 72 | 58 | 51 | 75 | 78 |

The Hungarian algorithm is to be used to minimise the cost of assigning workers to tasks.

**a** Explain why a dummy row is needed. **(1 mark)**

**b** Reducing rows first, apply the algorithm to obtain an allocation that minimises the total cost. **(8 marks)**

**c** State, with a reason, which task or tasks will not be completed. **(2 marks)**

## 2.3 Maximum profit allocation

The Hungarian algorithm finds the allocation of workers to tasks that **minimises** the total cost. You can modify the matrix so that you can use the same algorithm to find an allocation that **maximises** the total cost.

- **If you need to find an allocation of workers to tasks that maximises total cost, choose the largest value in the cost matrix, and subtract every entry in the cost matrix from that value.**

## Example 5

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| **Cherry** | 12 | 23 | 15 | 40 |
| **Jimmy** | 14 | 21 | 17 | 20 |
| **Mac** | 13 | 22 | 20 | 30 |
| **Plum** | 14 | 24 | 13 | 10 |

The numbers represent profits, in pounds, so that the profit made when Cherry does task $W$ is £12. Find the allocation of tasks to people to maximise the total income.

Start off with

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| Cherry | −12 | −23 | −15 | −40 |
| Jimmy | −14 | −21 | −17 | −20 |
| Mac | −13 | −22 | −20 | −30 |
| Plum | −14 | −24 | −13 | −10 |

The Hungarian algorithm finds minimums, so here you have to make all the numbers in the table negative. The algorithm will find the minimum solution, i.e. the most negative one, which will correspond to the actual maximum solution.

Subtract the most negative number from each element. In this case subtract −40, so that you lose the negative signs:

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| Cherry | 28 | 17 | 25 | 0 |
| Jimmy | 26 | 19 | 23 | 20 |
| Mac | 27 | 18 | 20 | 10 |
| Plum | 26 | 16 | 27 | 30 |

**Problem-solving**

You can start with this step, by choosing the largest value in the original matrix, 40, then subtracting every element in the cost matrix from this value.

Proceed as before.

The reduced matrix (after reducing both rows and columns) is

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| Cherry | 21 | 17 | 21 | 0 |
| Jimmy | 0 | 0 | 0 | 1 |
| Mac | 10 | 8 | 6 | 0 |
| Plum | 3 | 0 | 7 | 14 |

Three lines are needed to cover the zeros:

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| Cherry | 21 | 17 | 21 | 0 |
| Jimmy | 0 | 0 | 0 | 1 |
| Mac | 10 | 8 | 6 | 0 |
| Plum | 3 | 0 | 7 | 14 |

The minimum uncovered element is 3, so the next matrix is

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| Cherry | 18 | 17 | 18 | 0 |
| Jimmy | 0 | 3 | 0 | 4 |
| Mac | 7 | 8 | 3 | 0 |
| Plum | 0 | 0 | 4 | 14 |

Three lines are needed again to cover the zeros:

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| Cherry | 18 | 17 | 18 | 0 |
| Jimmy | 0 | 3 | 0 | 4 |
| Mac | 7 | 8 | 3 | 0 |
| Plum | 0 | 0 | 4 | 14 |

The minimum uncovered element is 3 again, so the next matrix is

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| Cherry | 15 | 14 | 15 | 0 |
| Jimmy | 0 | 3 | 0 | 7 |
| Mac | 4 | 5 | 0 | 0 |
| Plum | 0 | 0 | 4 | 17 |

Now four lines are needed to cover the zeros as shown.

The optimal solution is

Cherry – task $Z$ Jimmy – task $W$

Mac – task $Y$ Plum – task $X$

at a total profit of

40 + 14 + 20 + 24 = £98

## A useful tip

After the row and column reductions in the worked example above we had this matrix, which needed three lines to cover all the zeros.

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| **Cherry** | 21 | 17 | 21 | 0 |
| **Jimmy** | 0 | 0 | 0 | 1 |
| **Mac** | 10 | 8 | 6 | 0 |
| **Plum** | 3 | 0 | 7 | 14 |

We chose the lines like this:

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| **Cherry** | 21 | 17 | 21 | 0 |
| **Jimmy** | 0 | 0 | 0 | 1 |
| **Mac** | 10 | 8 | 6 | 0 |
| **Plum** | 3 | 0 | 7 | 14 |

This pattern gave the minimum uncovered element as 3, and another iteration was needed to locate the solution.

Instead, consider this pattern of lines:

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| **Cherry** | 21 | 17 | 21 | 0 |
| **Jimmy** | 0 | 0 | 0 | 1 |
| **Mac** | 10 | 8 | 6 | 0 |
| **Plum** | 3 | 0 | 7 | 14 |

In this case, the minimum uncovered element is 6 and we get the following matrix:

| | Task $W$ | Task $X$ | Task $Y$ | Task $Z$ |
|---|---|---|---|---|
| **Cherry** | 15 | 11 | 15 | 0 |
| **Jimmy** | 0 | 0 | 0 | 7 |
| **Mac** | 4 | 2 | 0 | 0 |
| **Plum** | 3 | 0 | 7 | 20 |

This also gives the optimal solution, but with one fewer iteration.

So if there is a choice in the pattern of lines, without increasing the number of lines, it makes sense to choose the pattern that leaves the minimum uncovered element as large as possible, as this is likely to lead to a speedier solution.

## Exercise 2C

**In questions 1 to 3**, the tables show the **profit**, in pounds, of allocating workers to tasks.

Reducing rows first, use the Hungarian algorithm to find an allocation that maximises the profit. You should make your method clear, show the table at each stage and state your final solution and its profit.

**1**

| | Task $C$ | Task $D$ | Task $E$ |
|---|---|---|---|
| **Worker $L$** | 37 | 15 | 12 |
| **Worker $M$** | 25 | 13 | 16 |
| **Worker $N$** | 32 | 41 | 35 |

2

| | Task $S$ | Task $T$ | Task $U$ | Task $V$ |
|---|---|---|---|---|
| Worker $C$ | 36 | 34 | 32 | 35 |
| Worker $D$ | 37 | 32 | 34 | 33 |
| Worker $E$ | 42 | 35 | 37 | 36 |
| Worker $F$ | 39 | 34 | 35 | 35 |

3

| | Task $E$ | Task $F$ | Task $G$ | Task $H$ |
|---|---|---|---|---|
| Worker $R$ | 20 | 22 | 14 | 24 |
| Worker $S$ | 20 | 19 | 12 | 20 |
| Worker $T$ | 13 | 10 | 18 | 16 |
| Worker $U$ | 22 | 23 | 9 | 28 |

E/P **4** In a strategy video game, a player must assign five units $A$, $B$, $C$, $D$ and $E$ to five tasks.

Each unit must be assigned to at most one task and each task must be done by just one unit. The number of points scored when each unit is assigned to each task is shown in the table.

| | Farm | Research | Build | Mine | Explore |
|---|---|---|---|---|---|
| Unit $A$ | 85 | 95 | 86 | 87 | 97 |
| Unit $B$ | 110 | 111 | 95 | 115 | 100 |
| Unit $C$ | 90 | 95 | 86 | 93 | 105 |
| Unit $D$ | 85 | 87 | 84 | 85 | 87 |
| Unit $E$ | 100 | 100 | 105 | 120 | 95 |

The Hungarian algorithm is to be used to find the allocation of units to tasks which will maximise the total number of points scored.

**a** Explain how this table must be modified before the algorithm can be implemented. **(1 mark)**

**b** Use the Hungarian algorithm to obtain an allocation which maximises the number of points scored, and state the number of points. **(8 marks)**

E **5** Workers $A$, $B$, $C$, $D$ and $E$ are to be assigned to tasks $V$, $W$, $X$, $Y$ and $Z$. Each worker is assigned to just one task and each task is to be completed by just one worker.

The table shows the profit made, in pounds, by allocating each worker to each task.

| | $V$ | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|---|
| $A$ | 74 | 68 | 123 | 86 | 95 |
| $B$ | 79 | 75 | 111 | 92 | 93 |
| $C$ | 68 | 72 | 134 | 96 | 99 |
| $D$ | 78 | 67 | 109 | 88 | 92 |
| $E$ | 70 | 78 | 130 | 81 | 88 |

**a** Reducing rows first, use the Hungarian algorithm to find the allocation of workers to tasks that provides the maximum profit. **(8 marks)**

**b** State the maximum value of the profit. **(2 marks)**

## 2.4 Managing incomplete data

You can adapt the Hungarian algorithm to manage incomplete data.

- **If it is not possible to assign a task to a given person, you can enter a large value into the matrix at the appropriate place. This makes these particular assignments 'unattractive'.**

### Example 6

An agency needs to assign four relief chefs Denis, Hilary, Robert and Trudy to four restaurants serving Chinese, French, Indian and Italian food. The travelling expenses, in pounds, that will be paid to each chef are shown in the table. Some of the chefs cannot work in some of the restaurants, since they are not familiar with that style of cookery.

| | New Dell (Chinese) | Aye Full Tower (French) | Hows a Curry (Indian) | Piece a Pasta (Italian) |
|---|---|---|---|---|
| **Denis** | – | 27 | 15 | 40 |
| **Hilary** | 14 | 21 | 17 | 13 |
| **Robert** | 20 | – | 13 | – |
| **Trudy** | 14 | 24 | 10 | 30 |

Use the Hungarian algorithm, reducing rows first, to obtain an allocation that minimises the total travelling expenses paid.

| | New Dell (Chinese) | Aye Full Tower (French) | Hows a Curry (Indian) | Piece a Pasta (Italian) |
|---|---|---|---|---|
| Denis | 100 | 27 | 15 | 40 |
| Hilary | 14 | 21 | 17 | 13 |
| Robert | 20 | 100 | 13 | 100 |
| Trudy | 14 | 24 | 10 | 30 |

Reducing rows gives:

| | New Dell (Chinese) | Aye Full Tower (French) | Hows a Curry (Indian) | Piece a Pasta (Italian) |
|---|---|---|---|---|
| Denis | 85 | 12 | 0 | 25 |
| Hilary | 1 | 8 | 4 | 0 |
| Robert | 7 | 87 | 0 | 87 |
| Trudy | 4 | 14 | 0 | 20 |

**Problem-solving**

Replace each forbidden allocation with a large number. This number is usually found by (at least) doubling the largest entry. In this case, enter values of 100 for each forbidden allocation.

Now continue as usual, applying the Hungarian algorithm to the new matrix.

Reducing columns gives:

| | New Dell (Chinese) | Aye Full Tower (French) | Hows a Curry (Indian) | Piece a Pasta (Italian) |
|---|---|---|---|---|
| Denis | 84 | 4 | 0 | 25 |
| Hilary | 0 | 0 | 4 | 0 |
| Robert | 6 | 79 | 0 | 87 |
| Trudy | 3 | 6 | 0 | 20 |

The zeros can be covered by two lines like this:

| | New Dell (Chinese) | Aye Full Tower (French) | Hows a Curry (Indian) | Piece a Pasta (Italian) |
|---|---|---|---|---|
| Denis | 84 | 4 | 0 | 25 |
| Hilary | 0 | 0 | 4 | 0 |
| Robert | 6 | 79 | 0 | 87 |
| Trudy | 3 | 6 | 0 | 20 |

The minimum uncovered element is 3, so the matrix becomes

| | New Dell (Chinese) | Aye Full Tower (French) | Hows a Curry (Indian) | Piece a Pasta (Italian) |
|---|---|---|---|---|
| Denis | 81 | 1 | 0 | 22 |
| Hilary | 0 | 0 | 7 | 0 |
| Robert | 3 | 76 | 0 | 84 |
| Trudy | 0 | 3 | 0 | 17 |

Using the short cut,

- add 3 to the cells covered by two lines
- leave the cells covered by one line unchanged
- subtract 3 from the cells uncovered by a line

The zeros can now be covered with just three lines, like this:

| | New Dell (Chinese) | Aye Full Tower (French) | Hows a Curry (Indian) | Piece a Pasta (Italian) |
|---|---|---|---|---|
| Denis | 81 | 1 | 0 | 22 |
| Hilary | 0 | 0 | 7 | 0 |
| Robert | 3 | 76 | 0 | 84 |
| Trudy | 0 | 3 | 0 | 17 |

The smallest uncovered element is 1, so the matrix becomes

| | New Dell (Chinese) | Aye Full Tower (French) | Hows a curry (Indian) | Piece a Pasta (Italian) |
|---|---|---|---|---|
| Denis | 81 | 0 | 0 | 21 |
| Hilary | 1 | 0 | 8 | 0 |
| Robert | 3 | 75 | 0 | 83 |
| Trudy | 0 | 2 | 0 | 16 |

Four lines are needed to cover the zeros, so we have an optimal solution.

Denis – Aye Full Tower    Hilary – Piece a Pasta

Robert – Hows a curry    Trudy – New Dell

The total cost = 27 + 13 + 13 + 14 = £67

The method for managing incomplete data can be adapted to deal with **maximising** a profit.

## Example 7

A team of four workers $A$, $B$, $C$ and $D$ are to be assigned to four tasks 1, 2, 3 and 4.

Each worker can only be assigned to one task and each task is to be completed by one worker.

Worker $B$ cannot be assigned to task 3.

The table shows the profit, in pounds, to be made by each worker on each task.

| | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| ***A*** | 47 | 68 | 125 | 52 |
| ***B*** | 39 | 72 | – | 60 |
| ***C*** | 48 | 63 | 140 | 63 |
| ***D*** | 37 | 75 | 132 | 68 |

**a** Reducing rows first, use the Hungarian algorithm to allocate workers to tasks so that the profit is **maximised**. Show the table at each stage.

**b** Work out the maximum profit.

**Problem-solving**

To maximise the profit with incomplete data, first subtract every number from the largest value in the table. Then enter a suitably large value in the empty cell.

The largest value in the table is 140.
Subtracting every value from 140, the table becomes

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| *A* | 93 | 72 | 15 | 88 |
| *B* | 101 | 68 | – | 80 |
| *C* | 92 | 77 | 0 | 77 |
| *D* | 103 | 65 | 8 | 72 |

First, set the table up to provide the maximum profit.

The largest value in the table now is 103.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| *A* | 93 | 72 | 15 | 88 |
| *B* | 101 | 68 | 300 | 80 |
| *C* | 92 | 77 | 0 | 77 |
| *D* | 103 | 65 | 8 | 72 |

Now deal with the incomplete data. A suitably large number to put in the cell B3 is 300.

Reducing rows:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| *A* | 78 | 57 | 0 | 73 |
| *B* | 33 | 0 | 232 | 12 |
| *C* | 92 | 77 | 0 | 77 |
| *D* | 95 | 57 | 0 | 64 |

Reducing columns and covering the zeros:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| *A* | 45 | 57 | 0 | 61 |
| *B* | 0 | 0 | 232 | 0 |
| *C* | 59 | 77 | 0 | 65 |
| *D* | 62 | 57 | 0 | 52 |

Only two lines are needed to cover the zeros, so the solution is not optimal.
The smallest uncovered number is 45.
Using the short cut method and covering the zeros:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| *A* | 0 | 12 | 0 | 16 |
| *B* | 0 | 0 | 277 | 0 |
| *C* | 14 | 32 | 0 | 20 |
| *D* | 17 | 12 | 0 | 7 |

Only three lines are needed to cover the zeros, so the solution is not optimal.
The smallest uncovered value is 7.
Applying the short cut method again and covering the zeros:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| *A* | 0 | 5 | 0 | 9 |
| *B* | 7 | 0 | 284 | 0 |
| *C* | 14 | 25 | 0 | 13 |
| *D* | 17 | 5 | 0 | 0 |

Four lines are needed to cover the zeros, so the solution is optimal.

*A* – task 1 (£47) *C* – task 3 (£140)
*B* – task 2 (£72) *D* – task 4 (£68)

The maximum profit is
£47 + £72 + £140 + £68 = £327

## Exercise 2D

**In questions 1 to 3**, the tables show the cost, in pounds, of allocating workers to tasks.

The crosses '✗' indicate that that worker cannot be assigned to that task.

Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, showing the table at each stage and state your final solution and its cost.

**1**

| | Task *L* | Task *M* | Task *N* |
|---|---|---|---|
| Worker *P* | 48 | 34 | ✗ |
| Worker *Q* | ✗ | 37 | 67 |
| Worker *R* | 53 | 43 | 56 |

**2**

| | Task *D* | Task *E* | Task *F* | Task *G* |
|---|---|---|---|---|
| Worker *R* | 38 | 47 | 55 | 53 |
| Worker *S* | 32 | ✗ | 47 | 64 |
| Worker *T* | ✗ | 53 | 43 | ✗ |
| Worker *U* | 41 | 48 | 52 | 47 |

**3**

| | Task *P* | Task *Q* | Task *R* | Task *S* |
|---|---|---|---|---|
| Worker *A* | 46 | 53 | 67 | 75 |
| Worker *B* | 48 | ✗ | 61 | 78 |
| Worker *C* | 42 | 46 | 53 | 62 |
| Worker *D* | 39 | 50 | ✗ | 73 |

E/P **4** Five construction workers $J$, $K$, $L$, $M$ and $N$ are to be assigned to five tasks $R$, $S$, $T$, $U$ and $V$. The tasks all require the same piece of machinery, so will be carried out one after the other. Each worker must be assigned to at most one task and each task must be done by just one worker.

The time taken, in minutes, for each worker to complete each task is shown in the table below. Some workers are not qualified to carry out some tasks, as indicated in the table by a cross.

| | Task $R$ | Task $S$ | Task $T$ | Task $U$ | Task $V$ |
|---|---|---|---|---|---|
| **Worker $J$** | 143 | 112 | 149 | 137 | ✗ |
| **Worker $K$** | 149 | 106 | 153 | 115 | 267 |
| **Worker $L$** | 137 | 109 | 143 | 121 | ✗ |
| **Worker $M$** | 157 | ✗ | ✗ | 134 | 290 |
| **Worker $N$** | 126 | 101 | 132 | 111 | 253 |

The Hungarian algorithm is to be used to determine the quickest possible allocation of workers to tasks.

**a** Explain how you can deal with the entries in the table marked with a cross. **(1 mark)**

**b** Reducing rows first, use the Hungarian algorithm to obtain an allocation which minimises the total time taken. You should state how each table is formed. **(8 marks)**

**Problem-solving**

'State how each table is formed' means that you need to state:

- by how much you are reducing each row and column
- how many lines can cover the zeros at each stage and whether your solution is optimal
- by what value you are augmenting the matrix at each stage

E/P **5** Workers $P$, $Q$, $R$, $S$ and $T$ are assigned to tasks 1, 2, 3, 4 and 5. Each worker is assigned to just one task and each task is to be completed by one worker.

Worker $P$ cannot do task 5 and worker $S$ cannot do task 3.

The table shows the profit made by each worker for completing each task.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $P$ | 25 | 42 | 38 | 52 | – |
| $Q$ | 43 | 37 | 29 | 46 | 55 |
| $R$ | 30 | 26 | 44 | 35 | 47 |
| $S$ | 36 | 41 | – | 40 | 53 |
| $T$ | 39 | 45 | 37 | 46 | 49 |

**a** Reducing rows first, find the allocation of workers to tasks that **maximises** the total profit. Show the table at each stage. **(9 marks)**

**b** State the value of the maximum profit. **(2 marks)**

**Challenge**

The cost, in pounds, of training each of five workers to use one of four machines is shown in the table below.

| | Machine $A$ | Machine $B$ | Machine $C$ | Machine $D$ |
|---|---|---|---|---|
| **Worker 1** | 700 | 500 | 1800 | 150 |
| **Worker 2** | 600 | 450 | 1100 | 220 |
| **Worker 3** | 850 | 700 | 1300 | 300 |
| **Worker 4** | 500 | 450 | 1400 | 280 |
| **Worker 5** | 350 | 400 | 1000 | 200 |

The company wishes to train exactly one worker on each machine, whilst minimising the total cost of training.

The Hungarian algorithm is to be used to allocate workers to machines. Describe how the cost matrix could be adapted, given that:

**a** worker 2 must be trained to use machine $D$

**b** worker 2 must be trained to use one of the machines

The company decides that because they have more workers than machines, two workers will be trained on one machine.

**c** State with a reason which machine should have two workers trained to use it, and describe how the cost matrix can be adapted to allocate workers in this instance.

## 2.5 Linear programming

A

You can formulate allocation problems as linear programming problems.

In an allocation problem, you match one worker to just one task and each task to just one worker.

You have just two options for the decision variables, either the worker is going to do the task or they are not.

You use **binary coding** to signal this, with 1 representing allocating that worker to the task and 0 representing not allocating them to the task.

**Links** You can consider the allocation problem as a version of the transportation problem, where the total stock at each supplier is 1, and the total demand at each depot is 1. ← Section 1.5

- **There is a standard way of presenting an allocation problem as a linear programming problem:**
  - **Define your decision variables.**
  - **Write down the objective function.**
  - **Write down the constraints.**

**Watch out** You must include all of these elements when formulating a linear programming problem.

## Example 8

A

Four workers Amy, Bob, Chris and Dave are to be assigned to tasks 1, 2, 3 and 4. Each worker is assigned to just one task and each task is to be completed by one worker.

The time, in minutes, taken by each worker to complete each task is shown in the table.

The total time taken is to be minimised.

| | Task 1 | Task 2 | Task 3 | Task 4 |
|---|---|---|---|---|
| **Amy** | 14 | 17 | 21 | 25 |
| **Bob** | 18 | 15 | 19 | 22 |
| **Chris** | 16 | 14 | 17 | 28 |
| **Dave** | 20 | 18 | 20 | 24 |

Formulate this as a linear programming problem.

The decision variables are of the form $x_{ij}$ where:

$x_{ij}$ represents worker $i$ being assigned to task $j$.

$i \in \{A, B, C, D\}$ $\quad$ $j \in \{1, 2, 3, 4\}$

$$x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$$

Make sure you clearly define your decision variables. The convention is to use single letters, or numbers, for the values of $i$ and $j$.

The objective is to minimise the total time.

Using $P$ for the total time, the objective function is:

$$\begin{aligned} \text{Minimise } P = {} & 14x_{A1} + 17x_{A2} + 21x_{A3} + 25x_{A4} \\ & + 18x_{B1} + 15x_{B2} + 19x_{B3} + 22x_{B4} \\ & + 16x_{C1} + 14x_{C2} + 17x_{C3} + 28x_{C4} \\ & + 20x_{D1} + 18x_{D2} + 20x_{D3} + 24x_{D4} \end{aligned}$$

Write the word 'Minimise' and define your objective function. The time, in minutes, taken by Amy for example is 14, 17, 21 or 25, depending on which task she is allocated to. This can be written as $14x_{A1} + 17x_{A2} + 21x_{A3} + 25x_{A4}$, since only one of $x_{A1}, x_{A2}, x_{A3}, x_{A4}$ will have the value 1 and the others will have the value 0.

The constraints can be found as follows.

Since each worker is allocated just one task, the decision variables for each row will contain exactly one 1 and the rest will all be zeros.

Also, since each task is completed by just one worker, the decision variables for each column must also contain exactly one 1 and the rest will all be zeros.

Unlike with the standard transportation problem, you must allocate each worker to a task, and vice versa, so these constraints are equalities, not inequalities. ← Section 1.5

The constraints are:

$$\begin{aligned} x_{A1} + x_{A2} + x_{A3} + x_{A4} &= 1 \\ x_{B1} + x_{B2} + x_{B3} + x_{B4} &= 1 \\ x_{C1} + x_{C2} + x_{C3} + x_{C4} &= 1 \\ x_{D1} + x_{D2} + x_{D3} + x_{D4} &= 1 \\ x_{A1} + x_{B1} + x_{C1} + x_{D1} &= 1 \\ x_{A2} + x_{B2} + x_{C2} + x_{D2} &= 1 \\ x_{A3} + x_{B3} + x_{C3} + x_{D3} &= 1 \\ x_{A4} + x_{B4} + x_{C4} + x_{D4} &= 1 \end{aligned}$$

Make sure you give a constraint for each row and for each column. There should be 8 in total.

**Notation** You can also write these constraints using sigma notation:

$\sum x_{Aj} = 1, \sum x_{Bj} = 1, \sum x_{Cj} = 1, \sum x_{Dj} = 1,$

$\sum x_{i1} = 1, \sum x_{i2} = 1, \sum x_{i3} = 1, \sum x_{i4} = 1$

## Example 9

The table shows the time, in minutes, taken for four workers *A*, *B*, *C* and *D* to do each of three tasks *P*, *Q* and *R*. Each worker can only be assigned one task, and each task is to be completed by one worker.

Worker *B* cannot be assigned to task *R* and worker *D* cannot be assigned to task *Q*.

| | **P** | **Q** | **R** |
|---|---|---|---|
| **A** | 10 | 14 | 20 |
| **B** | 12 | 14 | – |
| **C** | 11 | 16 | 22 |
| **D** | 13 | – | 21 |

Formulate the problem as a linear programming problem, defining your variables and making the objective and constraints clear.

| | **P** | **Q** | **R** | **S** |
|---|---|---|---|---|
| **A** | 10 | 14 | 20 | 0 |
| **B** | 12 | 14 | 50 | 0 |
| **C** | 11 | 16 | 22 | 0 |
| **D** | 13 | 50 | 21 | 0 |

There are more workers than tasks, so introduce a dummy column.

Replace each forbidden allocation with a large number, in this case 50.

Define the decision variables:

Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$

where $i \in \{A, B, C, D\}$ and $j \in \{P, Q, R, S\}$

Write down the objective function:

$$\text{Minimise } C = 10x_{AP} + 14x_{AQ} + 20x_{AR} + 12x_{BP} + 14x_{BQ} + 50x_{BR} + 11x_{CP} + 16x_{CQ} + 22x_{CR} + 13x_{DP} + 50x_{DQ} + 21x_{DR}$$

Subject to

$\sum x_{Aj} = 1,\ \sum x_{Bj} = 1,\ \sum x_{Cj} = 1,\ \sum x_{Dj} = 1$

These constraints require each worker to be assigned to a task (or the dummy)

$\sum x_{iP} = 1,\ \sum x_{iQ} = 1, \sum x_{iR} = 1,\ \sum x_{iS} = 1$

These constraints require each task (including the dummy) to have exactly one worker assigned.

- **You can formulate maximisation problems as linear programming problems by maximising your objective function.**

## Example 10

Four machines 1, 2, 3 and 4 are to be used to perform four tasks *A*, *B*, *C* and *D*. Each machine is to be assigned to just one task and each task must be assigned to just one machine. The profit generated when each machine performs each task is given in the table. The company wishes to maximise the total profit.

| | **A** | **B** | **C** | **D** |
|---|---|---|---|---|
| **1** | 10 | 2 | 8 | 6 |
| **2** | 9 | 3 | 11 | 3 |
| **3** | 3 | 1 | 4 | 2 |
| **4** | 3 | 2 | 1 | 5 |

Formulate the above assignment problem as a linear programming problem, defining your variables and making the objective and constraints clear.

A

Define the decision variables:

Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$

where $i \in \{1, 2, 3, 4\}$ and $j \in \{A, B, C, D\}$

Write down the objective function:

$$\text{Maximise } C = 10x_{1A} + 2x_{1B} + 8x_{1C} + 6x_{1D} + 9x_{2A} + 3x_{2B} + 11x_{2C} + 3x_{2D} + 3x_{3A} + x_{3B} + 4x_{3C} + 2x_{3D} + 3x_{4A} + 2x_{4B} + x_{4C} + 5x_{4D}$$

Write down the constraints:

Subject to
$$x_{1A} + x_{1B} + x_{1C} + x_{1D} = 1$$
$$x_{2A} + x_{2B} + x_{2C} + x_{2D} = 1$$
$$x_{3A} + x_{3B} + x_{3C} + x_{3D} = 1$$
$$x_{4A} + x_{4B} + x_{4C} + x_{4D} = 1$$
$$x_{1A} + x_{2A} + x_{3A} + x_{4A} = 1$$
$$x_{1B} + x_{2B} + x_{3B} + x_{4B} = 1$$
$$x_{1C} + x_{2C} + x_{3C} + x_{4C} = 1$$
$$x_{1D} + x_{2D} + x_{3D} + x_{4D} = 1$$

**Problem-solving**

You are not applying the Hungarian algorithm to this allocation problem, so you do not need to adapt the cost matrix by subtracting values. Instead, your objective is to **maximise** the objective function, which is defined as the total profit.

**Notation** Alternatively, you could write these constraints as

$\sum x_{1j} = 1, \sum x_{2j} = 1, \sum x_{3j} = 1, \sum x_{4j} = 1,$
$\sum x_{iA} = 1, \sum x_{iB} = 1, \sum x_{iC} = 1, \sum x_{iD} = 1$

## Exercise 2E

E/P **1** Three workers $L$, $M$ and $N$ are to be assigned to three tasks $C$, $D$ and $E$.

Each worker can be assigned to at most one task and each task must be done by just one worker. The time, in minutes, that each worker takes to complete each task is shown in the table below.

| | **Task $C$** | **Task $D$** | **Task $E$** |
|---|---|---|---|
| **Worker $L$** | 37 | 15 | 12 |
| **Worker $M$** | 25 | 13 | 16 |
| **Worker $N$** | 32 | 41 | 35 |

The total time taken is to be minimised. Formulate the problem as a linear programming problem. You must define your decision variables and make your objective function and constraints clear. **(5 marks)**

A E/P **2** The cost, in hundreds of pounds, of training each of four workers to carry out four tasks is shown in the table below.

Each worker is to be trained in exactly one task, and each task must have one worker trained to carry it out.

| | Task $S$ | Task $T$ | Task $U$ | Task $V$ |
|---|---|---|---|---|
| **Worker $C$** | 36 | 34 | 32 | 35 |
| **Worker $D$** | 37 | 32 | 34 | 33 |
| **Worker $E$** | 42 | 35 | 37 | 36 |
| **Worker $F$** | 39 | 34 | 35 | 35 |

The total cost of training is to be minimised. Formulate this as a linear programming problem, defining your decision variables and making your objective function and constraints clear.

**(5 marks)**

E/P **3** Four workers $A$, $B$, $C$ and $D$ are to be assigned to three sites 1, 2 and 3 in order to collect the names and addresses of people who may be considering changing their fuel supplier. The number of names and addresses that each worker is likely to collect at each site is shown in the table to the right. Worker $A$ cannot travel to site 3.

| | 1 | 2 | 3 |
|---|---|---|---|
| $A$ | 11 | 15 | – |
| $B$ | 14 | 18 | 17 |
| $C$ | 16 | 13 | 23 |
| $D$ | 15 | 14 | 22 |

Each worker must be assigned to just one site and each site assigned to just one worker. The fuel company wishes to **maximise** the number of names and addresses collected.

Formulate this as a linear programming problem, defining your variables and making the objective and constraints clear.

E/P **4** Four workers $A$, $B$, $C$ and $D$ are to be assigned to tasks $W$, $X$, $Y$ and $Z$. Each worker is to be assigned to just one task and each task is to be completed by a single worker.

The table shows the profit, in hundreds of pounds, made by each worker for completing each task.

| | $W$ | $X$ | $Y$ | $Z$ |
|---|---|---|---|---|
| $A$ | 12 | 8 | 11 | 9 |
| $B$ | 14 | 10 | 9 | 13 |
| $C$ | 11 | 9 | 12 | 10 |
| $D$ | 13 | 11 | 10 | 12 |

**a** Explain how the table should be modified so that the Hungarian algorithm can be used to find the maximum total profit. **(1 mark)**

**b** Reducing rows first, use the Hungarian algorithm to obtain an allocation of workers to tasks which maximises the total profit. **(7 marks)**

**c** Formulate the above problem as a linear programming problem, defining your decision variables, stating your objective and giving all the necessary constraints. **(5 marks)**

**Problem-solving**

In part **c** you could formulate a **maximisation** problem based on the original cost matrix, or a **minimisation problem** based on your modified cost matrix from parts **a** and **b**.

## Mixed exercise 2

(E) **1** A museum is staging a special exhibition. It has been loaned exhibits from other museums and from private collectors. Seven days before the exhibition starts these exhibits will be arriving at the airport, road depot, docks and railway station and in each case the single load has to be transported to the museum. There are four local companies that could deliver the exhibits: Bring-it, Collect-it, Fetch-it and Haul-it. Since all four companies are helping to sponsor the exhibition, the museum wishes to use all four companies, allocating each company to just one arrival point.

The table shows the cost, in pounds, of using each company for each task.

| | Airport | Depot | Docks | Station |
|---|---|---|---|---|
| **Bring-it** | 322 | 326 | 326 | 328 |
| **Collect-it** | 318 | 325 | 324 | 325 |
| **Fetch-it** | 315 | 319 | 317 | 320 |
| **Haul-it** | 323 | 322 | 319 | 321 |

The museum wishes to minimise its transportation costs.

Reducing rows first, use the Hungarian algorithm to determine the allocation that minimises the total cost. You must make your method clear and show the table after each stage. State your final allocation and its cost. **(8 marks)**

(E/P) **2** A medley relay swimming team consists of four swimmers. The first member of the team swims one length of backstroke, then the second person swims a length of breaststroke, then the next a length of butterfly and finally the fourth person a length of crawl. Each member of the team must swim just one length. Each of the team members could swim any of the lengths, but some members of the team are faster at one or two particular strokes.

The table shows the time, in seconds, each member of the team took to swim each length using each type of stroke during the last training session.

| | Back | Breast | Butterfly | Crawl |
|---|---|---|---|---|
| **Jack** | 18 | 20 | 19 | 14 |
| **Kyle** | 19 | 21 | 19 | 14 |
| **Liam** | 17 | 20 | 20 | 16 |
| **Mike** | 20 | 21 | 20 | 15 |

**a** Use the Hungarian algorithm, reducing rows first, to find an allocation that minimises the total time it takes the team to complete all four lengths. **(8 marks)**

**b** State the best time in which this team could complete the race. **(1 mark)**

**c** Show that there is more than one way of allocating the team so that they can achieve this best time. **(2 marks)**

**Note** In fact there are four optimal solutions to this problem.

E/P **3** Five tour guides work at Primkal Mansion. They talk to groups of tourists about five particularly significant rooms. Each tour guide will be stationed in a particular room for the day, but may change rooms the next day. The tourists will listen to each talk before moving on to the next room. Once they have listened to all five talks they will head off to the gift shop.

The table shows the average length of each tour guide's talk in each room.

| | **Grand Hall** | **Dining Room** | **Gallery** | **Bedroom** | **Kitchen** |
|---|---|---|---|---|---|
| **Alf** | 8 | 19 | 11 | 14 | 12 |
| **Betty** | 12 | 17 | 14 | 18 | 20 |
| **Charlie** | 10 | 22 | 18 | 14 | 19 |
| **Donna** | 9 | 15 | 16 | 15 | 21 |
| **Eve** | 14 | 23 | 20 | 20 | 19 |

A tourist party arrives at the Mansion.

**a** Use the Hungarian algorithm, reducing rows first, to find the shortest time that the tour could take. You should state the optimal allocation and its length and show the state of the table at each stage. **(7 marks)**

**b** Adapt the table and reapply the Hungarian algorithm, reducing rows first, to find the longest time that the tour could take. You should state the optimal allocation and its duration and show the state of the table at each stage. **(8 marks)**

E/P **4** A company hires out chauffer-driven, luxury stretch-limousines. They have to provide a limousine for three events next Saturday night: an award ceremony, a film premiere and a celebrity party. The company has four chauffeurs available and the cost, in pounds, of assigning each of them to each event is shown in the table.

| | **Award ceremony** | **Film premiere** | **Celebrity party** |
|---|---|---|---|
| **Denzel** | 245 | 378 | 459 |
| **Eun-Ling** | 250 | 387 | 467 |
| **Frank** | 224 | 350 | 442 |
| **Gabby** | 231 | 364 | 453 |

The company wishes to minimise its total costs.

**a** Explain why it is necessary to add a dummy event. **(1 mark)**

**b** Reducing rows first, use the Hungarian algorithm to determine the allocation that minimises the total cost. You should state the optimal allocation and its cost and show the state of the table at each stage. **(7 marks)**

E/P **5** A large office block is to be serviced and supplied by six companies Blue Supplies, Green Services, Orange Office Supplies, Red Co, Teal Services and Yellow Ltd. These companies have each applied to take care of catering, cleaning, computer supplies/servicing, copying, postal services and maintenance. The table shows the daily cost of using each company, in pounds.

| | Catering | Cleaning | Computer | Copying | Post | Maintenance |
|---|---|---|---|---|---|---|
| **Blue** | No | 863 | 636 | 628 | 739 | 634 |
| **Green** | 562 | 796 | 583 | 478 | 674 | No |
| **Orange** | No | 825 | 672 | 583 | 756 | 710 |
| **Red** | 635 | 881 | 650 | 538 | No | 685 |
| **Teal** | 688 | 934 | No | 554 | No | 742 |
| **Yellow** | 624 | 835 | 580 | No | 712 | No |

For political reasons the owners of the office block will use all six companies, one for each of the six tasks.

Some of the companies cannot offer some services and this is indicated by 'No'.

Use the Hungarian algorithm, reducing rows first, to allocate the companies to the servies in such a way as to minimise the total cost. You should state the optimal allocation and its cost and show the state of the table at each stage. **(9 marks)**

E/P **6** The owners of a theme park wish to provide a café, coffee shop, restaurant and snack shop at four sites: next to the ghost train, log flume, roller coaster and Teddie's Adventure. They employ a market researcher who estimates the daily profit of each type of catering at each site. The estimated daily profit, in pounds, is shown in the following table.

The market researcher also suggests that some types of catering are not suitable at some of the sites. These are indicated by 'No'.

| | Café | Coffee shop | Restaurant | Snack shop |
|---|---|---|---|---|
| **Ghost train** | 834 | 365 | 580 | 648 |
| **Log flume** | 874 | 375 | No | 593 |
| **Roller coaster** | 743 | 289 | No | 665 |
| **Teddie's Adventure** | 899 | 500 | 794 | No |

Using the Hungarian algorithm, determine the allocation that provides the maximum daily profit. **(9 marks)**

**Problem-solving**

This question is a maximising question and one with incomplete data. You need to write a large number at the sites marked 'No', so that they become 'unattractive' to the algorithm, after you have altered it to look for the maximum solution.

E/P **7** Six workers $A$, $B$, $C$, $D$, $E$ and $F$ are to be assigned to five tasks 1, 2, 3, 4 and 5. Each worker can only be assigned one task and each task is to be completed by one worker.

Worker $B$ cannot do task 1 and worker $D$ cannot do task 5.

The table shows the profit, in pounds, earned by each worker on completion of each task.

| | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| ***A*** | 128 | 142 | 153 | 133 | 155 |
| ***B*** | – | 138 | 147 | 139 | 147 |
| ***C*** | 135 | 144 | 144 | 130 | 158 |
| ***D*** | 141 | 156 | 154 | 142 | – |
| ***E*** | 150 | 141 | 157 | 145 | 160 |
| ***F*** | 132 | 149 | 140 | 140 | 157 |

Use the Hungarian algorithm to allocate workers to tasks to maximise the total profit. State the value of the profit. **(9 marks)**

A E/P **8** Krunchy Cereals Ltd will send four salesmen $P$, $Q$ $R$ and $S$ to visit four store managers at $A$, $B$, $C$ and $D$ to take orders for their new products. Each salesman will visit only one store manager and each store manager will be visited by just one salesman. The expected value, in thousands of pounds, of the orders won is shown in the table. The company wishes to maximise the value of the orders.

| | ***A*** | ***B*** | ***C*** | ***D*** |
|---|---|---|---|---|
| ***P*** | 13 | 17 | 15 | 18 |
| ***Q*** | 15 | 19 | 12 | 19 |
| ***R*** | 16 | 20 | 13 | 22 |
| ***S*** | 14 | 15 | 17 | 14 |

Formulate this as a linear programming problem, defining your variables and making the objective and constraints clear. **(6 marks)**

E/P **9** Four workers $P$, $Q$, $R$ and $S$ are to be assigned to four tasks 1, 2, 3 and 4. Each worker is to be assigned to one task and each task must be assigned to one worker. The cost, in pounds, of using each worker for each task is given in the table below.

| | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| ***P*** | 143 | 243 | 247 | 475 |
| ***Q*** | 132 | 238 | – | 437 |
| ***R*** | 126 | 207 | 197 | 408 |
| ***S*** | – | 222 | 238 | 445 |

Worker $Q$ cannot do task 3 and worker $S$ cannot do task 1.

The total cost is to be minimised.

Formulate this as a linear programming problem, defining your variables and making the objective and constraints clear. **(6 marks)**

**Challenge**

Four workers $W$, $X$, $Y$ and $Z$ are to be assigned to eight tasks $A$, $B$, $C$, $D$, $E$, $F$, $G$ and $H$. The table shows the cost, in pounds, for each worker to complete each task.

| | ***A*** | ***B*** | ***C*** | ***D*** | ***E*** | ***F*** | ***G*** | ***H*** |
|---|---|---|---|---|---|---|---|---|
| ***W*** | 35 | 41 | 28 | 52 | – | 51 | 74 | 48 |
| ***X*** | 51 | 39 | 40 | 55 | 42 | 50 | 63 | 54 |
| ***Y*** | 38 | 45 | 39 | 50 | 48 | 47 | 65 | 50 |
| ***Z*** | 47 | – | 48 | 51 | 45 | 53 | 64 | 52 |

Worker $W$ is unable to do task $E$ and worker $Z$ is unable to do task $B$.

Individual workers may be allocated to more than one task, but no worker can do more than three tasks. The total cost is to be minimised.

Formulate this as a linear programming problem.

## Summary of key points

1 In allocation problems there must be the same number of tasks as workers.

2 To reduce a cost matrix:
- subtract the least value in each row from each element of that row
- using the new matrix, subtract the least value in each column from each element in that column

3 If it is possible to allocate tasks to workers in such a way that the corresponding entries in the reduced cost matrix are all zero, then this allocation is optimal.

4 **The Hungarian algorithm**
- Find the reduced cost matrix. Reduce rows first.
- Find the minimum number of straight lines (horizontal or vertical) which will cover all of the zeros in the matrix.
- In an $n \times n$ matrix, if you cannot cover the zeros in fewer than $n$ lines, you have an optimal solution and you stop.
- If you can cover the zeros in fewer than $n$ lines, the solution can be improved.
- Draw in the lines and look for the smallest uncovered element, $e$.
- Add $e$ to the elements in each covered row and each covered column, adding it twice to any element covered twice.
- Subtract $e$ from every element in the matrix.
- Repeat procedure until an optimal solution is found.

5 If you have to solve an allocation problem which is not $n \times n$, you can introduce dummy rows or dummy columns with zero entries.

6 If you need to find an allocation of workers to tasks that maximises total cost, choose the largest value in the cost matrix, and subtract every entry in the cost matrix from that value.

7 If it is not possible to assign a task to a given person, you can enter a large value into the matrix at the appropriate place. This makes these particular assignments 'unattractive'.

A

8 There is a standard way of presenting an allocation problem as a **linear programming** problem:
- define your decision variables
- write down the objective function
- write down the constraints

9 You can formulate maximisation problems as linear programming problems by maximising your objective function.

# 3 Flows in networks 1

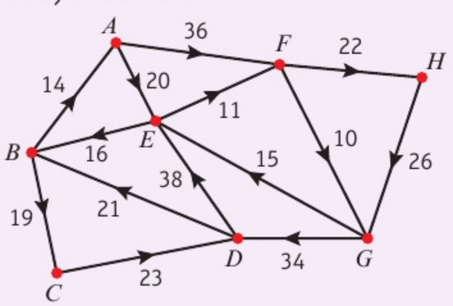
## Objectives

After completing this chapter you should be able to:

- Understand and analyse flow through a network → pages 73–108
- Find initial flows in networks → pages 85–88
- Use the labelling procedure to augment a flow to determine the maximum flow in a network → pages 88–98
- Use the maximum flow – minimum cut theorem to prove that a flow is maximal → pages 98–103

## Prior knowledge check

This diagram represents a road network. The number on each arc represents the capacity of the road to carry traffic. This is given by the maximum number of vehicles that can travel along the road in one minute. Traffic moves along each road in the direction shown by the arrow.




**a** Locate the points where traffic jams are likely to occur.

**b** Explain why a traffic jam may occur at $D$ even though $34 + 23 < 38 + 21$

When designing road networks, road planners must take into account the maximum number of vehicles that each road can handle in a given time. If the flow of vehicles towards a junction is greater than the flow of vehicles away from the junction then this will produce a traffic jam. → Mixed exercise Q5

## 3.1 Flows in networks

Network flow problems deal with **capacitated directed networks**. A capacitated directed network is one in which each arc has:

- an arrow indicating the permitted direction of flow
- a weight indicating the **capacity** of that arc. This is the maximum amount of flow that can pass along that arc.

■ **A vertex, $S$, is called a source if all arcs connected to $S$ are directed away from $S$. A vertex, $T$, is called a sink if all arcs connected to $T$ are directed towards $T$.**

Capacitated directed networks can be used to model real-life situations in which the flow of units between two particular points might be restricted. Examples include:

- Water flow in a water supply or sewage network
- Flow of people along corridors or paths
- Movement of cars on a road network
- Routing data packets through a computer network

**Notation** Such networks may also be called **capacitated directed graphs**, or **capacitated digraphs**.

**Links** In this chapter, arcs will have a single weight indicating their maximum capacity, and networks will have one source and one sink. A-level students may have to deal with arcs containing both upper and lower capacities, and with networks containing more than one source or more than one sink. → Chapter 4

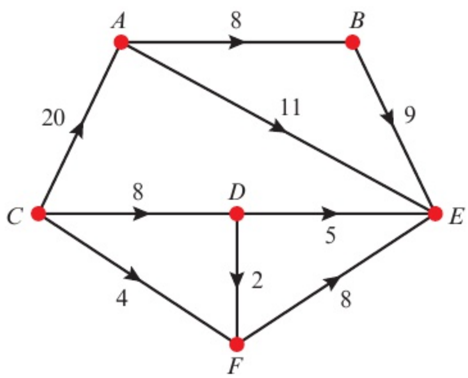
### Example 1

The diagram shows a capacitated directed network.

**a** Write down the capacity of arcs $DF$ and $BE$.

**b** State the source vertex.

**c** State the sink vertex.




**a** $DF$ has capacity 2 and $BE$ has a capacity of 9.

This means that the maximum flow along $DF$ is 2, from $D$ to $F$, and the maximum flow along $BE$ is 9, in the direction from $B$ to $E$.

**b** The source vertex is $C$.

All the arcs that include $C$ have arrows that are directed away from $C$.

**c** The sink vertex is $E$.

All the arcs that include $E$ have arrows that are directed towards $E$.

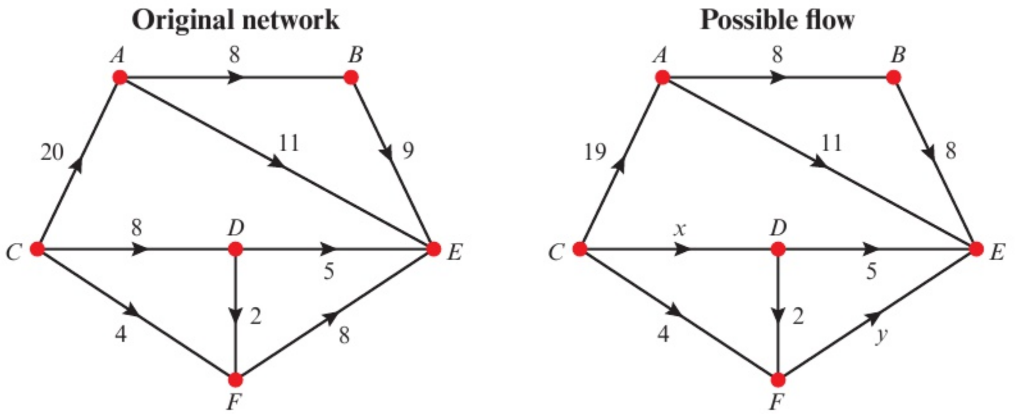
- **To show a flow through a network, assign a (non-negative) number to each arc so that it satisfies two conditions:**
  - **the feasibility condition, which says that the flow along each arc must not exceed the capacity of that arc.**
  - **the conservation condition on all but source and sink vertices, which says that**

    **the total flow into a vertex = the total flow out of the vertex**

    **so the flow cannot 'build up' at a vertex.**
- **If the arc contains a flow equal to its capacity we say that the arc is saturated.**
- **The value of a flow is the sum of the flows along all arcs leaving a source vertex. This is also equal to the sum of the flows along all arcs entering a sink vertex.**

## Example 2

The diagrams below show a possible flow through a capacitated directed network.




**a** Find the values of $x$ and $y$, explaining your reasoning.

**b** List the five saturated arcs.

**c** Write down the value of the flow.

**d** What is the current flow along route $CAE$?

**a** Flow into $D$ = flow out of $D$

$x = 2 + 5$ so $x = 7$

Flow into $F$ = flow out of $F$

$4 + 2 = y$ so $y = 6$

**b** The saturated arcs are $AB$, $AE$, $CF$, $DE$ and $DF$.

**c** The flow leaving $C$ (the source)

$= 19 + x + 4 = 19 + 7 + 4 = 30$

**d** From $C$ to $A$ the flow is 19. From $A$ to $E$ the flow is 11. So the flow along the route $CAE$ is 11.

Neither $D$ nor $F$ are source or sink vertices, so the conservation condition must be satisfied.

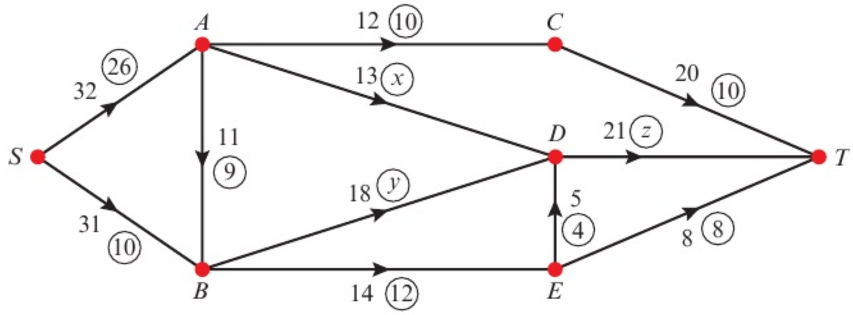
Look for arcs where the flow is equal to the capacity.

You could also calculate the value of the flow by looking at the flow arriving at $E$ (the sink).
$8 + 11 + 5 + y = 8 + 11 + 5 + 6 = 30$

When asked to find the flow along a particular route you need to find the maximum value that can pass along the entire route. In this case, although 19 can start along the given route, only 11 can continue along $AE$, so only 11 can travel along this route.

If you need to draw one diagram showing the directed network, together with a feasible flow, you can indicate the flow using circled numbers.

**Example 3**




The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern.

**a** State a saturated arc.

**b** Find the values of $x$, $y$ and $z$, explaining your reasoning.

**c** State the value of the initial flow.

**d** Write down:

  **i** the capacity of arc $BE$

  **ii** the current flow along arc $BE$.

**e** Find the current flow along $SABET$.

**a** $ET$ is saturated, because its flow is equal to its capacity.

**b** Flow into $A$ = flow out of $A$

$$26 = 10 + x + 9$$
$$x = 7$$

Flow into $B$ = flow out of $B$

$$10 + 9 = y + 12$$
$$y = 7$$

Flow into $D$ = flow out of $D$

$$x + y + 4 = z$$
$$7 + 7 + 4 = z$$
$$z = 18$$

**c** The feasible flow is 36.

**d** **i** The capacity of arc $BE$ is 14.

**ii** The current flow along arc $BE$ is 12.

**e** Along $SABET$ the current flow is 8.

Consider either the flow leaving $S$ or the flow arriving at $T$.

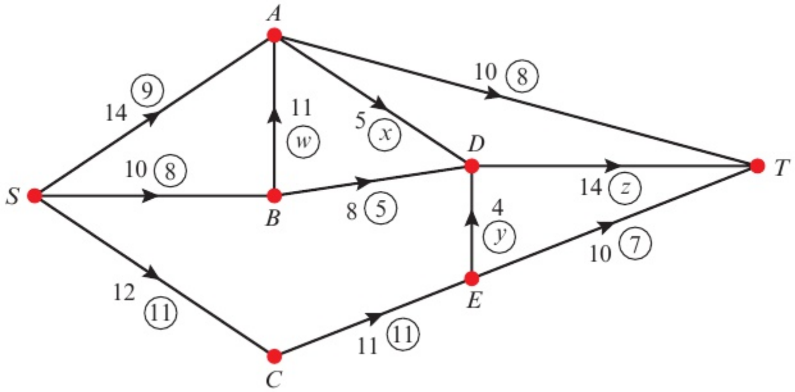
**Watch out** In the exam many candidates get the flow and the capacity along an arc confused. The capacity is the maximum permitted flow along an arc; the flow is the current value passing through the arc.

From $S$ to $A$ the flow is 26,
from $A$ to $B$ it is 9,
from $B$ to $E$ it is 12,
and from $E$ to $T$ it is 8.
The smallest of these is 8, so only 8 can pass along the entire route.

## Exercise 3A

**Answer templates for questions marked * are available at www.pearsonschools.co.uk/d2maths**

**1**




The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern.

**a** Find the values of $w$, $x$, $y$ and $z$, explaining your reasoning.

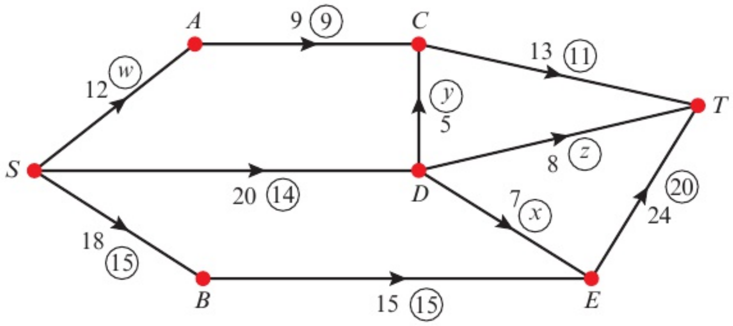
**b** State the value of the initial flow.

**c** Identify two saturated arcs.

**d** Write down the capacity of arc $BD$.

**e** What is the current flow along route $SAT$?

**2**




The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern.

**a** Find the values of $w$, $x$, $y$ and $z$, explaining your reasoning.

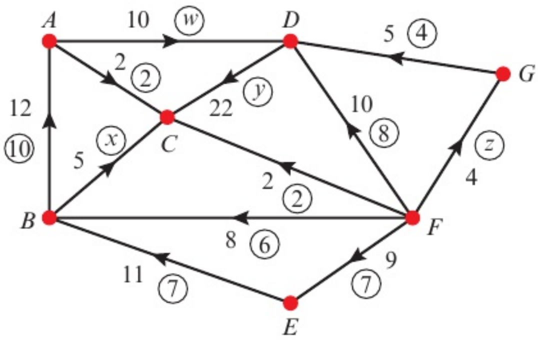
**b** State the value of the initial flow.

**c** Identify two saturated arcs.

**d** Write down the flow along arc $SD$.

**e** What is the current flow along route $SBET$?

**3**




The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern.

**a** State the source vertex.

**b** State the sink vertex.

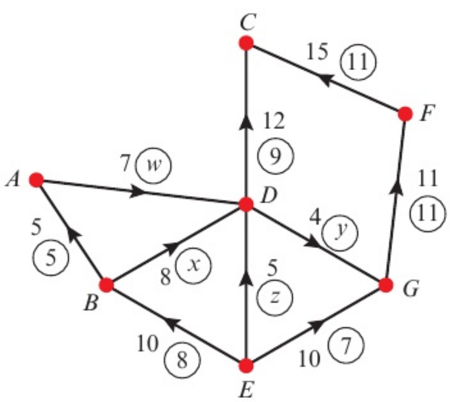
**c** Find the values of $w$, $x$, $y$ and $z$, explaining your reasoning.

**d** State the value of the feasible flow.

**e** Identify three saturated arcs.

**f** Write down the capacity of arc $FB$.

**4**




The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern.

**a** State the source vertex.

**b** State the sink vertex.

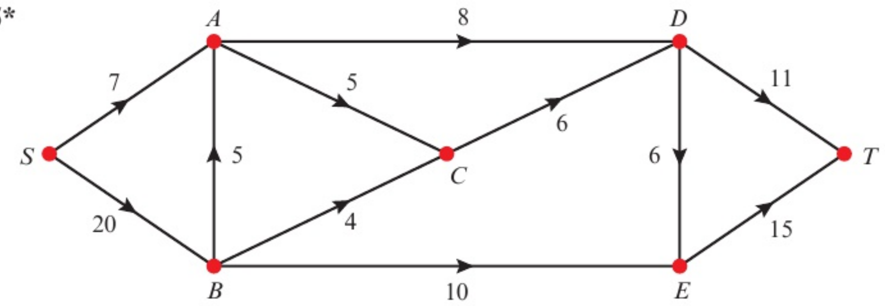
**c** Find the values of $w$, $x$, $y$ and $z$, explaining your reasoning.

**d** State the value of the initial flow.

**e** Identify four saturated arcs.

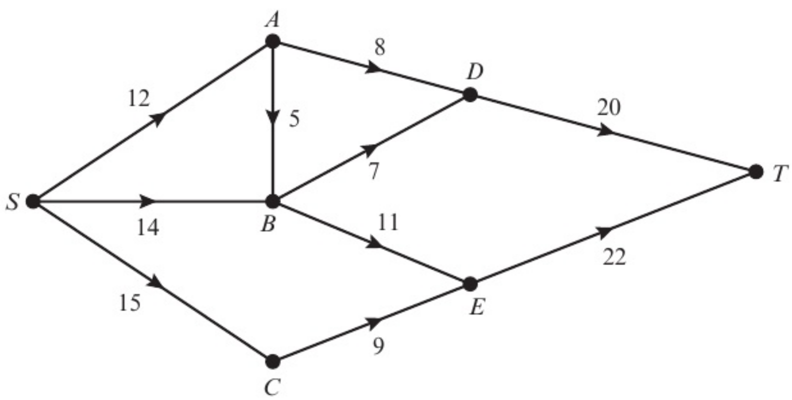
**f** Write down the flow along arc $FC$.

(P) **5***




The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. Find a feasible flow of at least 20 through the network from $S$ to $T$.

(E/P) **6*** A group of tourists on a guided tour leave point $S$ and traverse paths in a park to get to point $T$.




**a** State an assumption about the paths that has been made in order to model this situation using a directed network. **(1 mark)**

**b** Find a feasible flow that will allow 32 tourists per minute to leave point $S$ and arrive at point $T$. **(4 marks)**

**c** For this flow, write down the number of tourists each minute that take route $SBET$. **(2 marks)**

## 3.2 Cuts and their capacities

In network flow problems, it is common to seek the flow with the maximum possible value for a given network. This is called the **maximal flow**. In order to find this, you need to consider **cuts** in networks.

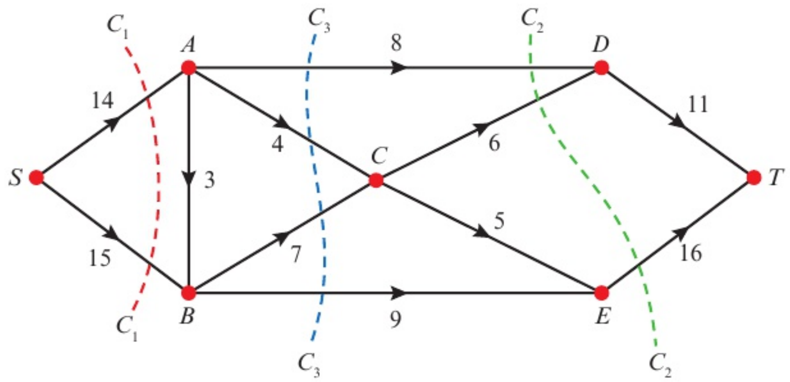
- **A cut, in a network with source $S$ and sink $T$, is a set of arcs whose removal separates the network into two parts $X$ and $Y$, where $X$ contains at least $S$ and $Y$ contains at least $T$.**

A cut divides the network into two. One piece **must** contain the source(s) and the other **must** contain the sink(s).

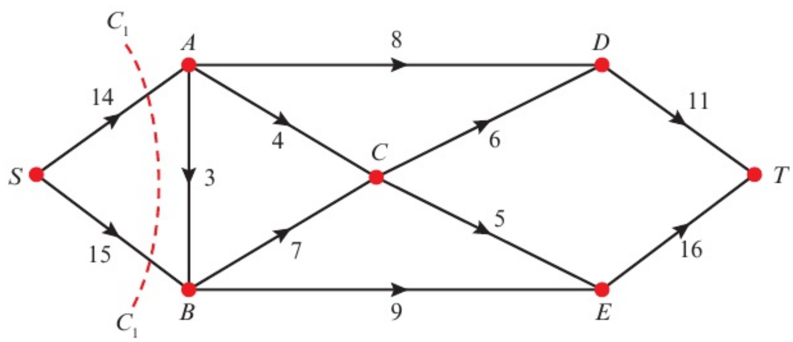
- **The capacity (value) of a cut is the sum of the capacities of those arcs in the cut which are directed from $X$ to $Y$.**

**Watch out** Do not confuse a cut (which is a set of arcs) with its capacity (which is a single value). Make sure you use the **capacities** of the arcs when calculating the capacity of a cut, and not the flow on each arc.

This diagram shows a capacitated directed network, together with three different cuts, labelled $C_1$, $C_2$ and $C_3$.

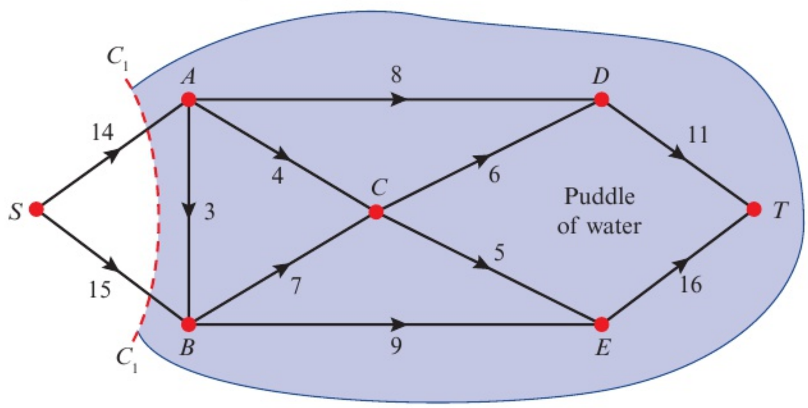



Cut $C_1$ breaks the network into two pieces like this.




It creates two **vertex sets** $\{S\}$ and $\{A, B, C, D, E, T\}$. These sets list which vertices are now in which 'broken half' of the network.

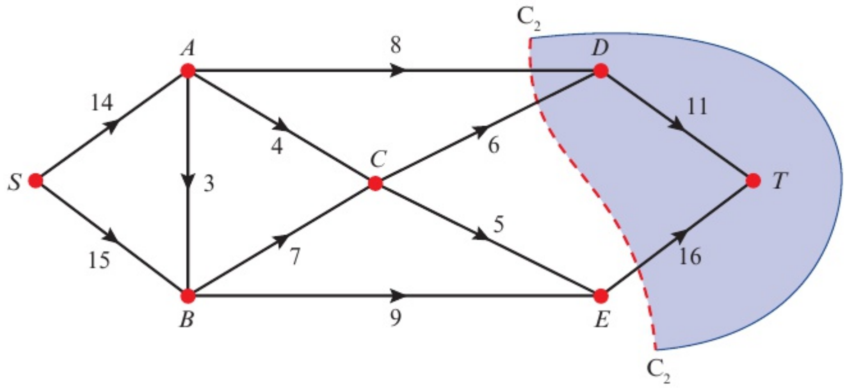
To evaluate the capacity of the cut it might help to imagine that the network represents water pipes, and that the pipes in the cut have been broken. This causes the water that would have run through those pipes to spill onto the floor, creating a puddle. Draw the puddle by connecting the top and bottom of the cut by drawing a line **behind** the sink.




To find the capacity of the cut you need to ask 'how much water is flowing into the puddle?' We assume that the broken pipes are flowing at capacity. So you have 14 coming from $SA$ and 15 from $SB$, giving a total of 29.

Capacity of cut $C_1 = 14 + 15 = 29$

'Drawing the puddle' for cut $C_2$ we get

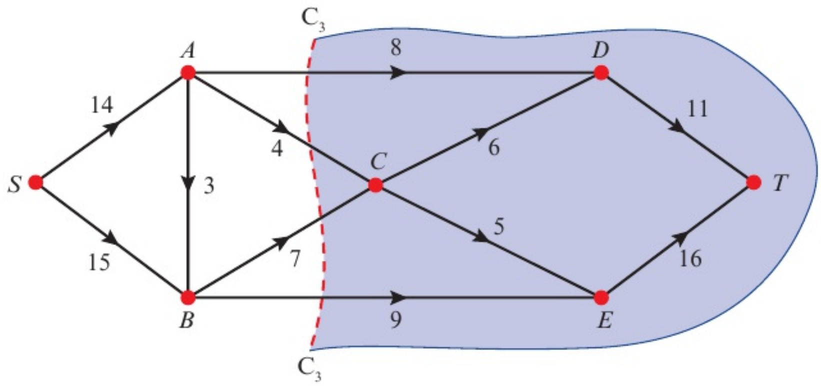



Flowing 'into the puddle' are $AD$ value 8, $CD$ value 6 and $ET$ value 16.

So the capacity of the cut is 8 + 6 + 16 = 30

Capacity of cut $C_2$ = 8 + 6 + 16 = 30

'Drawing the puddle' for cut $C_3$ we get




Flowing 'into the puddle' are $AD$ value 8, $AC$ value 4, $BC$ value 7 and $BE$ value 9.

So the capacity of the cut is 8 + 4 + 7 + 9 = 28

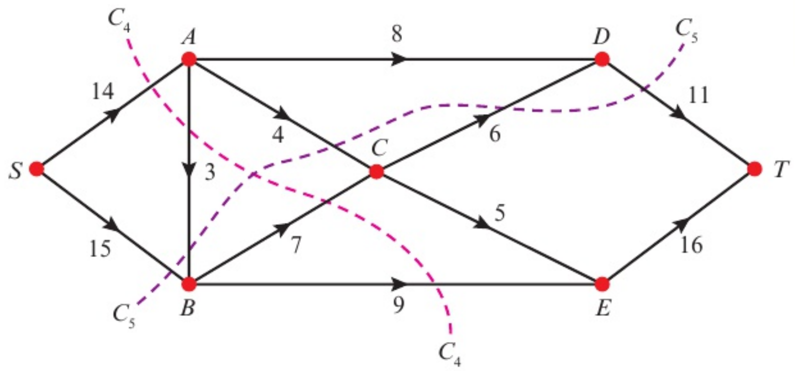
Capacity of cut $C_3$ = 8 + 4 + 7 + 9 = 28

- **When evaluating the capacity of a cut, only include the capacities of the arcs flowing into the cut.**
- **Arcs which are cut, but whose direction flows out of the cut, contribute zero to the capacity of the cut.**

Arcs directed from the source cut set to the sink cut set are said to flow **into** the cut. Those directed from the sink to source sets are said to flow **out** of the cut.

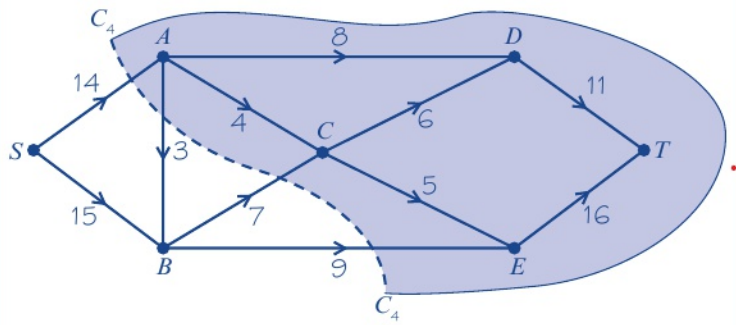
**Example 4**

Online Explore cuts and their capacities in this network using Geogebra.




The diagram shows the capacitated directed network from the previous page. The number on each arc represents the capacity of that arc.

Determine the values of cuts $C_4$ and $C_5$.



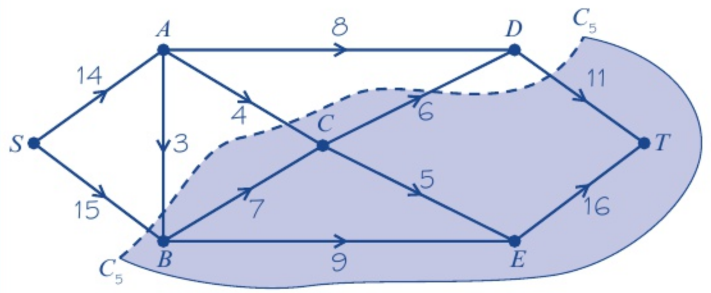

Value of cut $C_4 = 14 + 7 + 9 = 30$

Flowing into the cut are $SA$ value 14, $BC$ value 7 and $BE$ value 9.

If you look carefully at the arrow on $AB$ you can see it does not flow into the cut, so although $AB$ has been broken it contributes nothing to the value of the cut.

(In practical terms you can see that the supply to $A$ has been severed when you cut $SA$, so the 'pipe' $AB$ will be empty.)

So for this cut, only sum the capacities of arcs $SA$, $BC$ and $BE$.




Value of cut $C_5 = 11 + 4 + 3 + 15 = 33$

Flowing into the cut are $DT$ value 11, $AC$ value 4, $AB$ value 3 and $SB$ value 15.

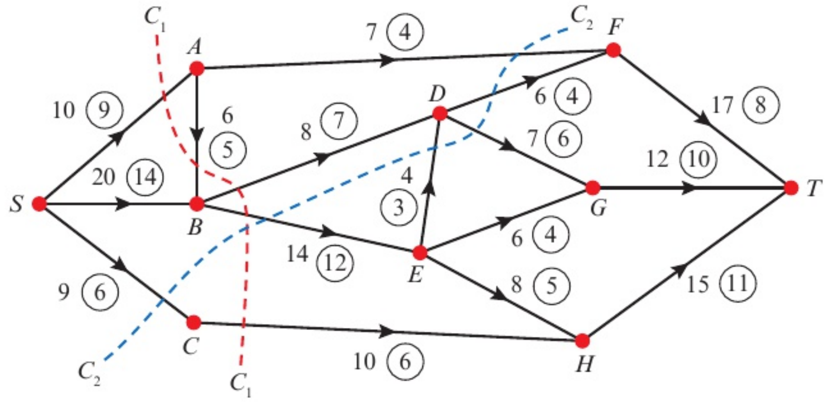
$CD$, although cut, does not contribute to the value of the cut, since its direction indicates that it flows out of the cut.

Notice also that you did not include arc $AB$ in cut $C_4$, but it has been included in cut $C_5$.

You will have noticed that the five cuts on the same network give four different values, ranging from 28 to 33. In general, different cuts give different values.

When a diagram is given that shows both capacities and a flow, the value of the cut is calculated using the **capacities**.

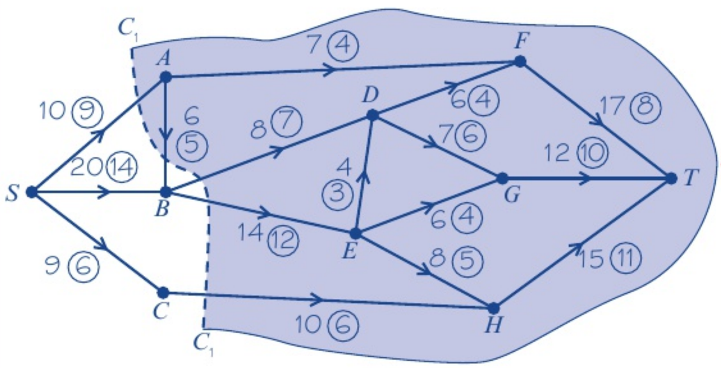
## Example 5




The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern.

Find the values of cuts $C_1$ and $C_2$.

Drawing cut $C_1$ we get




Flowing into the cut are arcs $SA$, $BD$, $BE$ and $CH$.

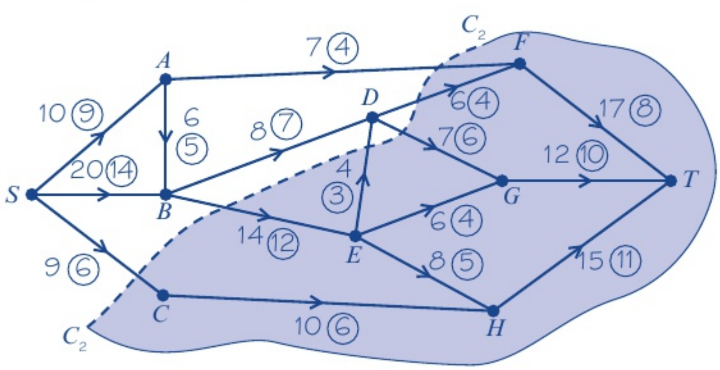
$AB$ flows out of the cut so contributes zero to its value.

So the value of the cut $= 10\ (SA) + 8\ (BD) + 14\ (BE) + 10\ (CH)$.

Value of cut $C_1 = 10 + 8 + 14 + 10 = 42$

**Watch out** Use the capacities to evaluate the cut, not the flow along each arc.

Drawing cut $C_2$ we get




Flowing into the cut are arcs $AF$, $DF$, $DG$, $BE$ and $SC$.

$ED$ flows out of the cut and so contributes zero to its value.

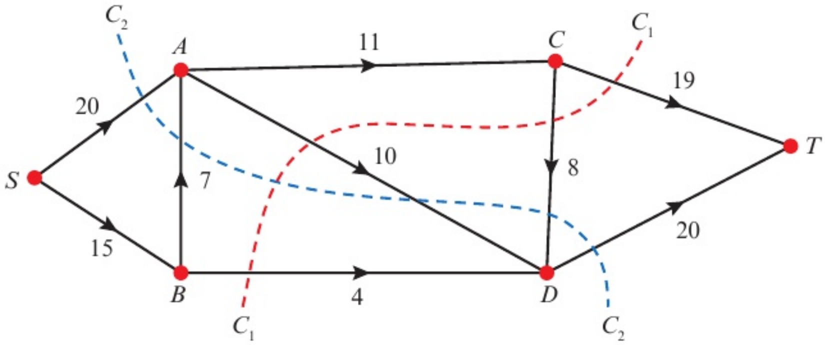
So the value of the cut $= 7\ (AF) + 6\ (DF) + 7\ (DG) + 14\ (BE) + 9\ (SC)$

Value of cut $C_2 = 7 + 6 + 7 + 14 + 9 = 43$

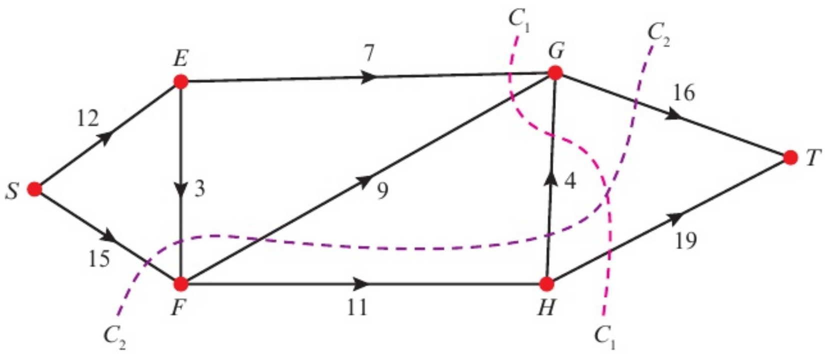
## Exercise 3B

**In questions 1 to 5**, the diagrams show capacitated directed networks. The number on each arc represents the capacity of that arc. Where shown, the numbers in circles represent an initial flow pattern. Evaluate the capacities of the cuts drawn.

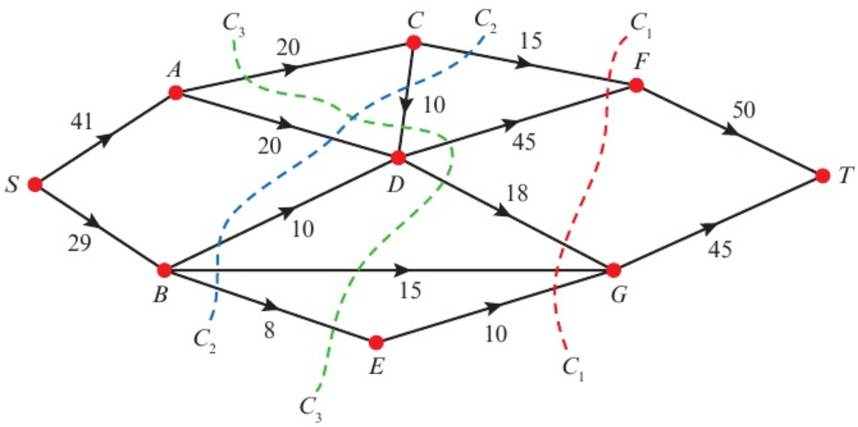
**1**



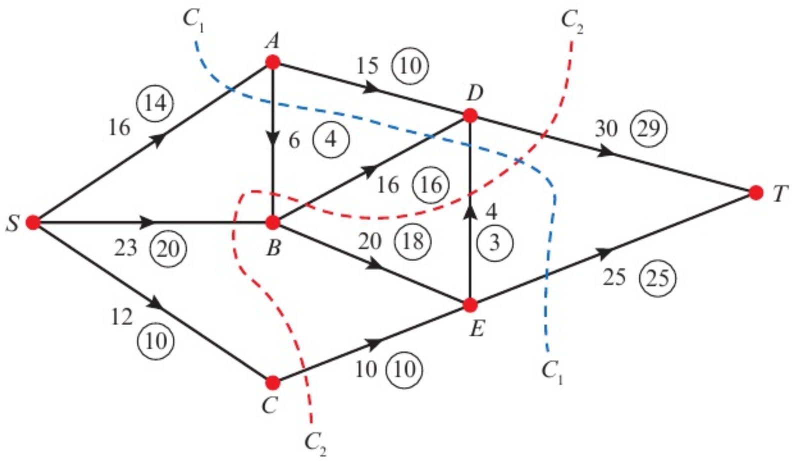

**2**



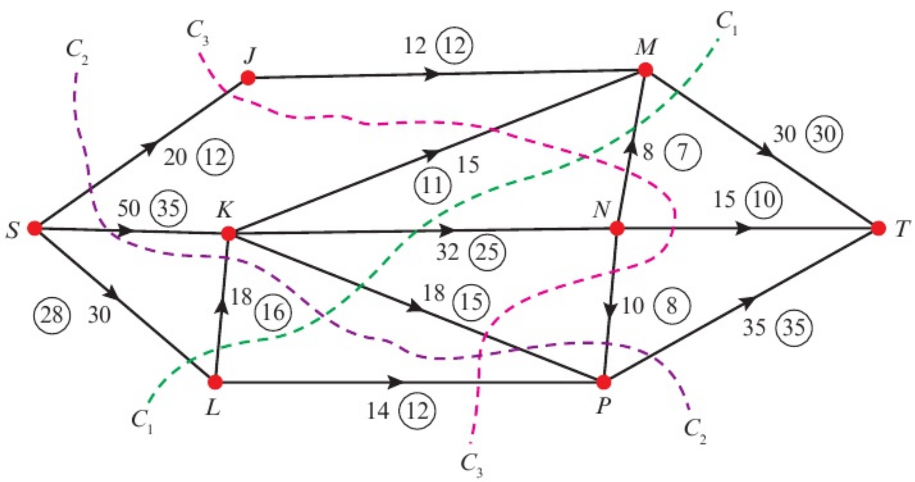

**3**



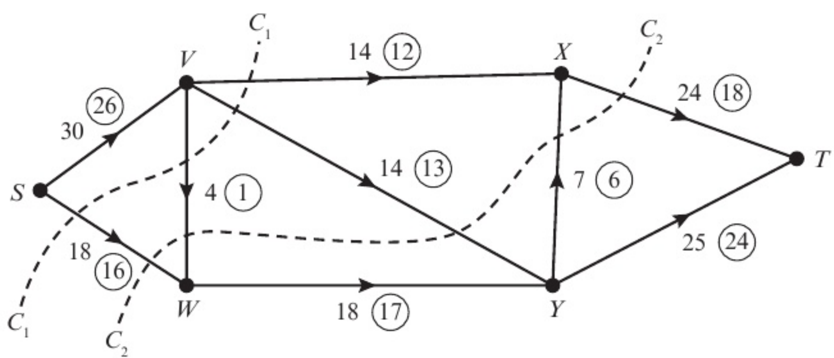
4




5




E **6 a** Define what is meant by a cut in a directed network. **(1 mark)**

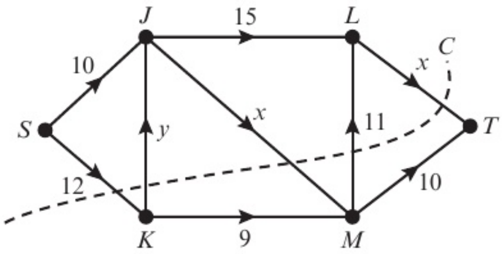



The diagram shows a flow in a directed network and two cuts. Find:

**b i** the value of the flow **(1 mark)**

**ii** the capacity of each cut. **(2 marks)**

E/P **7** The diagram shows a cut, $C$, with capacity 28 in a directed network.



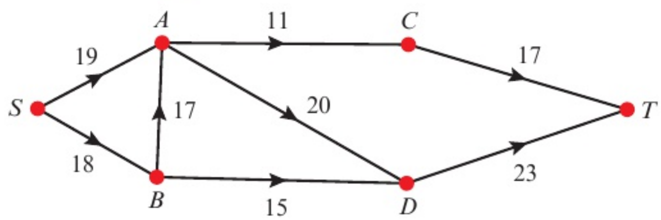

**a** Calculate the value of $x$. **(2 marks)**

**b** Explain why you do not have enough information to calculate the value of $y$. **(1 mark)**

## 3.3 Finding an initial flow

When looking for a maximal flow in a network, you need to start with an initial flow. In your exam you could be given an initial flow, or you may need to find one.

### Example 6




The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc.

**a** State the maximum flows along $SACT$ and $SBADT$.

**b** Show these on a diagram.

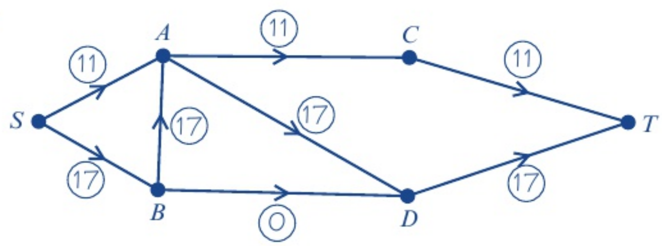
Using this as your initial flow pattern,

**c** calculate the value of the initial flow.

**a** The maximum flow along $SACT$ is 11.

The maximum flow along $SBADT$ is 17.

**b**



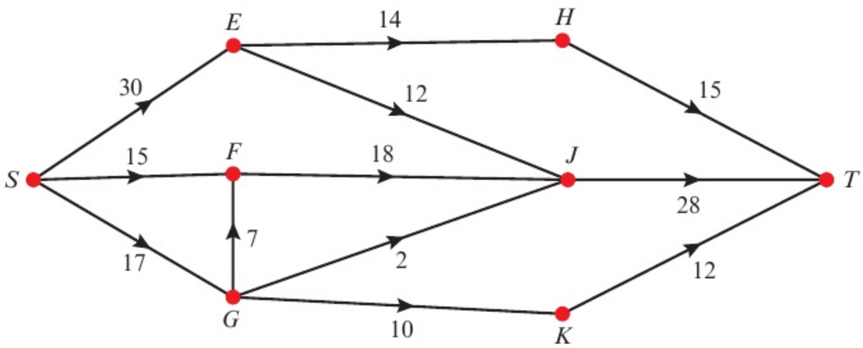

**c** The initial flow has a value of 28.

The arcs are $SA$ (19), $AC$ (11) and $CT$ (17). The greatest flow along that route is 11.

The arcs are $SB$ (18), $BA$ (17), $AD$ (20), $DT$ (23). The greatest flow possible is 17. Arc $BA$ has the lowest capacity and so this gives the maximum possible flow along this path.

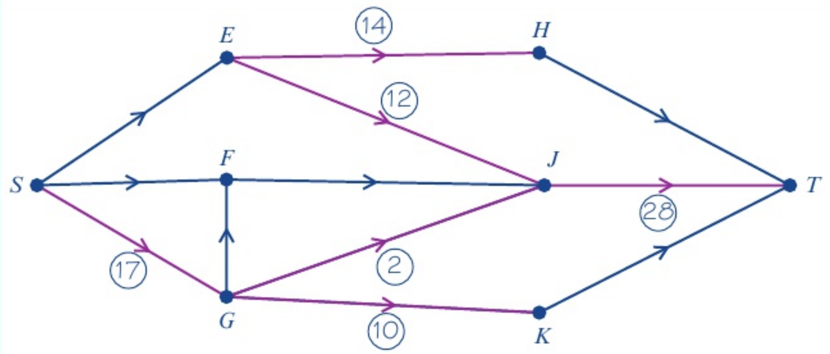
The total amount leaving $S$ (or arriving at $T$) is 28.

## Example 7

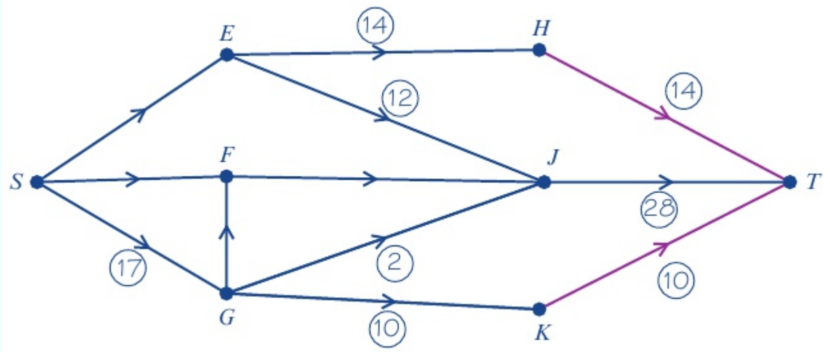



The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc.

Given that arcs $SG$, $EH$, $EJ$, $GJ$, $GK$ and $JT$ are saturated, draw an initial flow through the network.

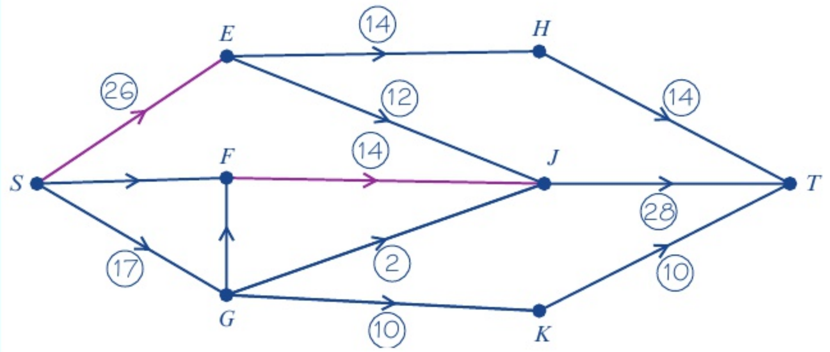



Start by noting the saturated arcs. Saturated means that these six arcs are at capacity, so you can place these six numbers onto the diagram.

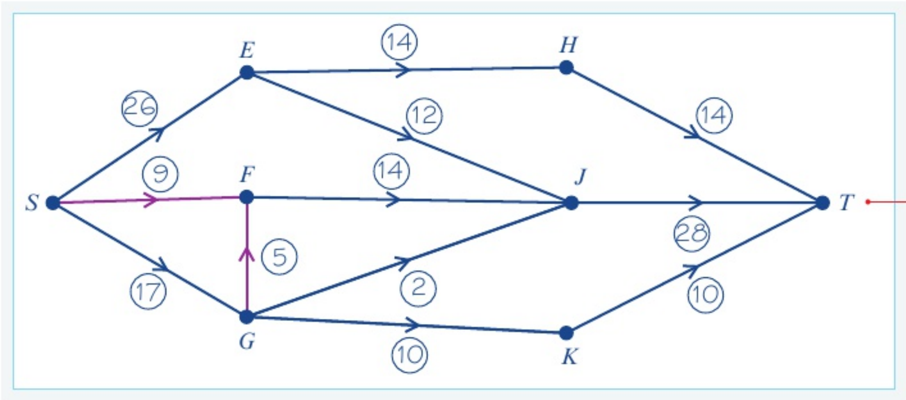



Now use flow-conservation to deduce some other values.

Look at the flow into $H$ and into $K$. You can add two more numbers.




Look at the flow into $J$ and into $E$ to add two more numbers.

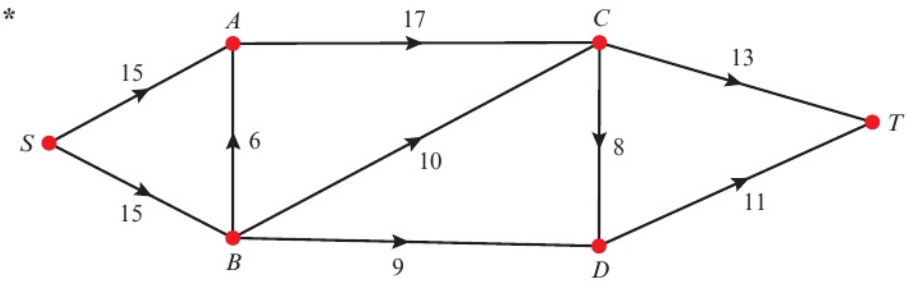



Finally, look at conserving the flow at $G$ and $F$. Since a flow of 52 is arriving at $T$, 52 must be leaving $S$.

## Exercise 3C

**Answer templates for questions marked * are available at www.pearsonschools.co.uk/d2maths**

**1***



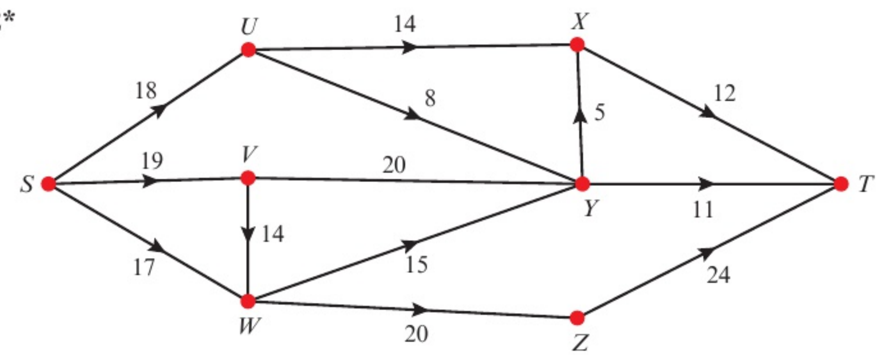

The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc.

**a** State the maximum flows along $SACT$ and $SBCDT$.

**b** Using these as an initial flow:

**i** show this flow on a diagram **ii** state the value of this flow.

**2***



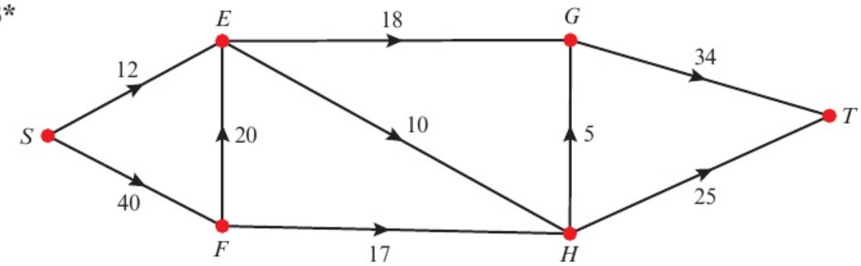

The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc.

**a** State the maximum flows along $SUXT$, $SWZT$ and $SVWYT$.

**b** Using these as an initial flow:

**i** show this flow on a diagram **ii** state the value of this flow.

**3***

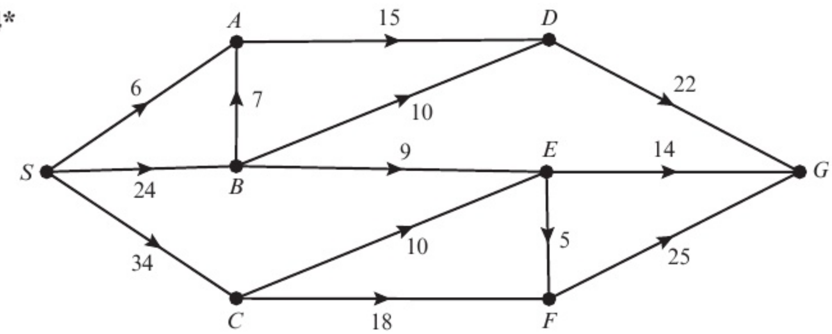



The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc.

**a** Given that arcs *SE*, *EG*, *EH*, *FH* and *HG* are saturated, draw an initial flow through the network.

**b** State the value of the initial flow.

E **4***




The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc.

An initial flow through the network is to be found.

**a** Explain why arc *FG* cannot be saturated. **(1 mark)**

**b** Given that arcs *SB*, *BA*, *BD*, *CE*, *CF*, *EF* and *DG* are saturated, draw an initial flow through the network. **(2 marks)**

**c** State the value of the initial flow. **(1 mark)**

## 3.4 Flow-augmenting routes

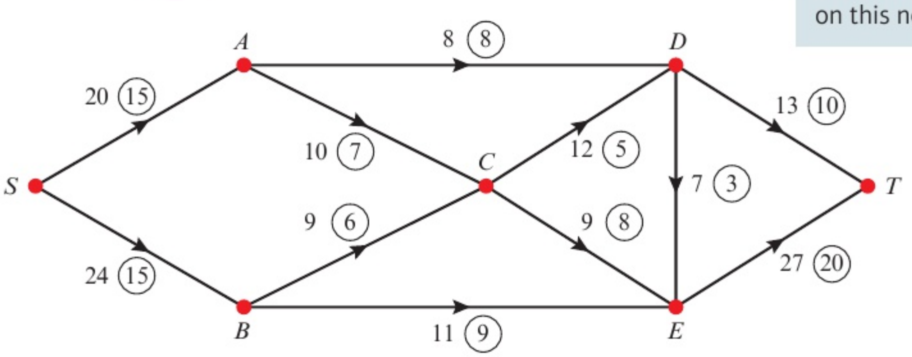
Although it is possible to find flows by inspection on small, simple diagrams, you need an algorithm that you can apply to more complicated networks that will guarantee finding the maximum possible flow through the network.

It may be necessary to re-route the initial flow, so you use the **labelling procedure** which allows you to reduce the flow along some arcs as well as to increase it in others. You use this to find **flow-augmenting routes** to increase the flow through the network.

- **In the labelling procedure, you draw two arrows on each arc.**
  - **The 'forward' arrow (the arrow in the same direction as the arc) identifies the amount by which the flow along that arc can be increased. (This is the spare capacity.)**
  - **The 'backward' arrow identifies the amount by which the flow in the arc could be reduced. (It shows the value of the current flow in that arc but not the direction.)**
  - **Each flow-augmenting route starts at $S$ and finishes at $T$.**
  - **You may use both forward and backward arrows to find a route, but there must be capacity in the direction in which you wish to move.**

**Example 8**

**Online** Find feasible flows on this network using Geogebra.



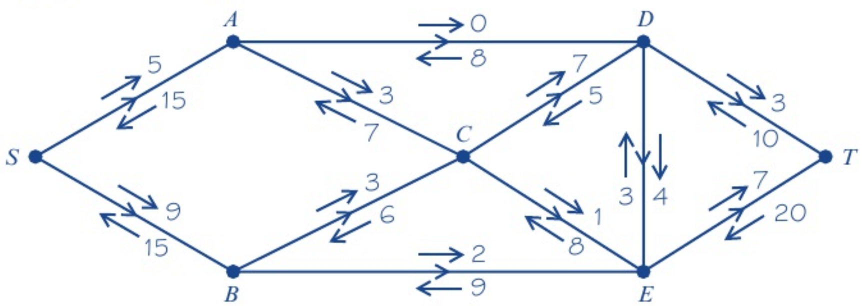

The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern.

Starting from this initial flow pattern,

**a** use the labelling procedure to obtain a maximum flow pattern through the network from $S$ to $T$, and explain why your flow is maximal. You should list each flow-augmenting route you use together with its flow.

**b** Draw your final flow pattern.

**a** Apply the labelling procedure.




The arrows going the in same direction as the arc in the original network record the spare capacity in each arc.

The arrows going in the opposite direction to that seen in the original diagram record the amount by which the flow could be reduced in each arc. They give the value, but not the direction of the flow currently in the arc.

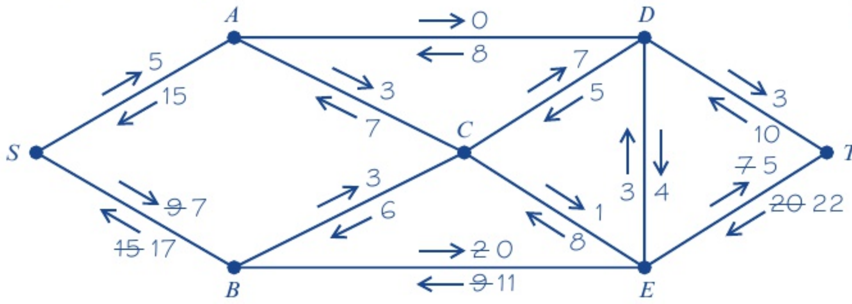
Starting at $S$, seek a route to $T$. You can use the arrows in either direction, but there must be spare capacity in the direction you wish to move.

You could send an additional flow of 2 along route $SBET$. ($BE$ has maximum spare capacity of 2.)

There are many different routes that could be chosen.

Make sure you write out each flow augmenting route together with its additional flow.

The updated diagram looks like this.




Notice that you have **added 2** to the number on each **backward** arrow along the route and **subtracted 2** from the number on each **forward** arrow along the route.

Along $SB$ there were 9 'spaces' left for extra flow, but you have now used 2 of them up, leaving just 7 for the 'forward' number. You have increased the flow along that arc by 2 so the 'backward' number goes from 15 to 17.

Along $BE$ there were 2 'spaces' left for extra flow. You have used them both so there is no 'space' left and the forward arrow decreases from 2 to 0. The flow along arc $BE$ has increased by 2 from 9 so the new backward flow is 11.

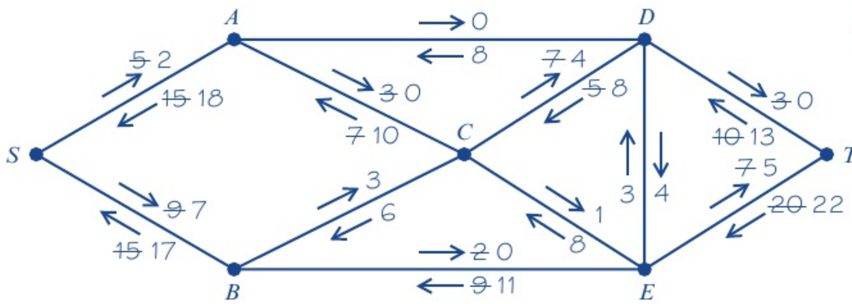
Along $ET$ there were 7 'spaces'. You have used 2 of them so the spaces for extra flow is now 5. The flow has increased by 2 from 20, so the new 'backward' flow is 22.

**Notation** We often refer to the arrows that go in the direction of the flow-augmenting route as **forward arrows**, and those going in the opposite direction as **backward arrows**.

The sense of 'forward' and 'backward' is determined by the flow-augmenting route, and not by the original direction of those arcs.

You could send an additional flow of 3 along $SACDT$.

The updated diagram looks like this.




**Hint** In your exam you do not need to draw each stage of the updated diagrams. You can just update your initial diagram. If you wish to cross out your old values, use a single neat line so that your working can be clearly seen.

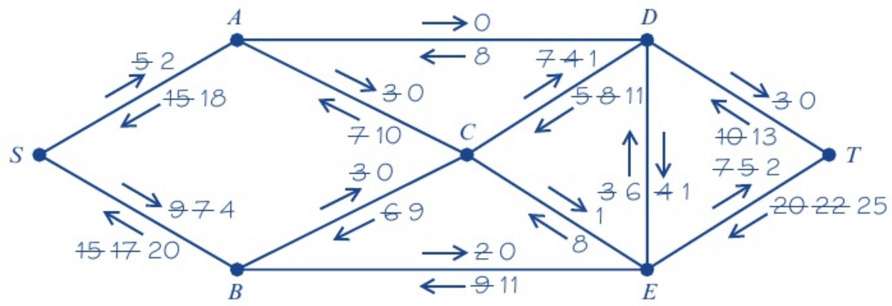
Now look for further routes.

There is no point in looking for routes beginning $SA$, because all arcs leaving $A$ are now saturated as their 'forward' arrows are zero, indicating that there is no spare capacity.

If you start at $SB$ you then you **must** go to $C$, since there is no other unsaturated exit from $B$. So your route must start with $SBC$. From $C$ you have a choice, $CD$ or $CE$. You may choose either. Arbitrarily you choose $SBCD$. You **must** then use $DE$ as there is no other unsaturated exit available from $D$. This gives the route as $SBCDET$.

The maximum extra flow along $SBCDET$ is 3.

The updated diagram looks like this.

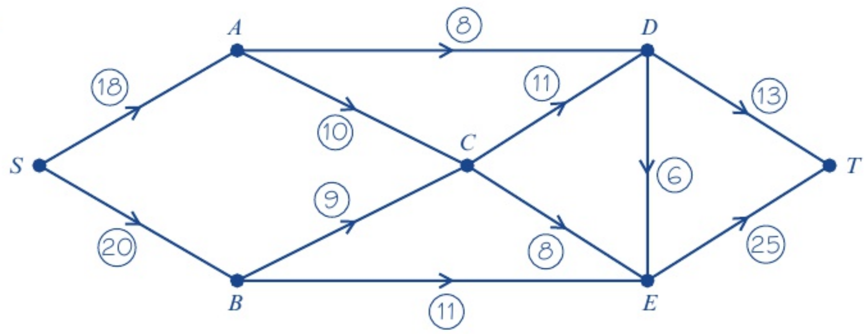


This is the last route. You cannot start at $SA$. All arcs leaving $A$ are now saturated as their forward numbers are zero.

You cannot start at $SB$, since all the arcs leaving $B$ have zero as their forward number, indicating that they too are saturated.

Thus there are no further flow-augmenting routes to find, and you have increased the flow through the network to its maximum.

b



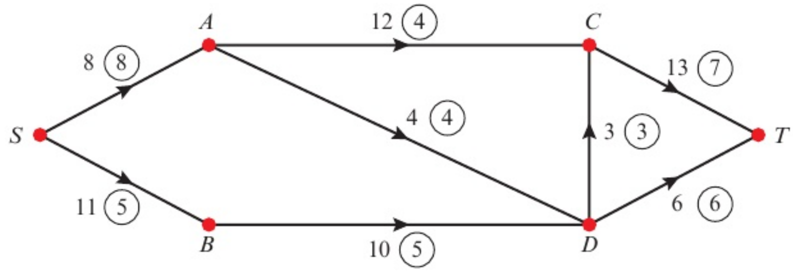
Make sure the arrows on your final flow diagram show the direction of each arc. The value of the flow along each arc will be equal to the final value next to the arrow which points in the **opposite** direction to the direction of the arc.

**Watch out** If you have applied the labelling procedure correctly the flow will satisfy the feasibility and conservation conditions. If it does not, double-check your working.

When using the labelling procedure to find a flow-augmenting route, you may move in either direction along an arc, as long as there is not a zero in the direction you wish to move.

- **When sending an extra flow of value *f* along a flow-augmenting route:**
  - **subtract *f* from the number on each 'forward' arrow on the route**
  - **add *f* to the number on each 'backward' arrow on the route.**
- **Sometimes a flow-augmenting route will reduce the flow along a particular arc.** **This is indicated by using the arrow that points in the opposite direction to the direction of the arc.**

## Example 9




The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern.

**a** Write down the value of the initial flow.

Starting from this initial flow pattern,

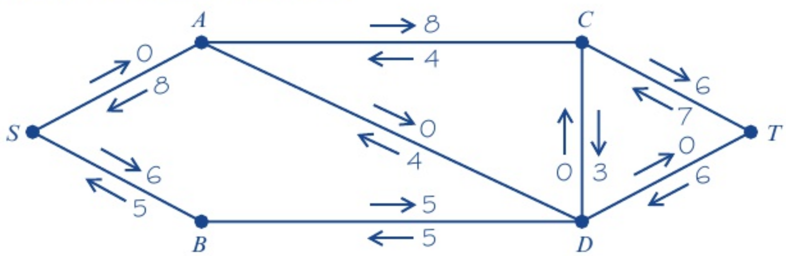
**b** use the labelling procedure to obtain a maximum flow pattern through the network from $S$ to $T$, listing each flow-augmenting route you use together with its flow.

**c** Draw your final flow pattern.

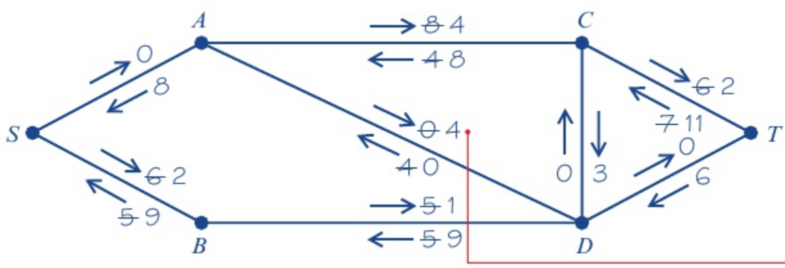
**d** State the value of the maximum flow through this network.

**a** The value of the initial flow is 13.

**b** Apply the labelling procedure:




You can see that $SA$ is saturated, because there is a zero in the direction $S$ to $A$. You must start with $SBD$.
From $D$ the arcs $DC$ and $DT$ have zero capacity in the direction you need to move, so your only exit is $DA$.
From $A$ you can go to $C$ and then on to $T$.
This gives your flow-augmenting route as $SBDACT$.
The smallest 'forward' number along this route is 4, giving the value of the additional flow.
The updated diagram is:



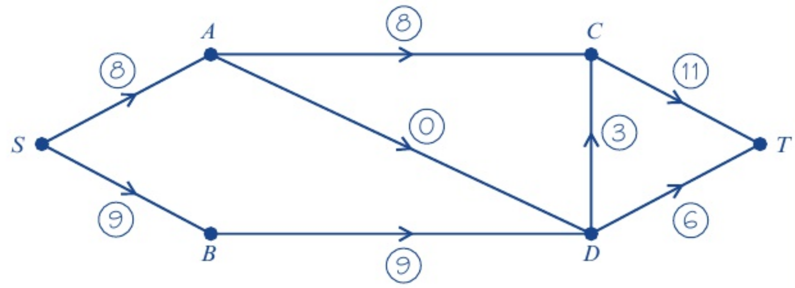

There are no further flow-augmenting routes you can use, so your flow is maximal.

Look at the numbers in circles. There is a flow of 5 + 8 leaving $S$ and a flow of 6 + 7 arriving at $T$.

**Watch out** This flow-augmenting route makes use of an arrow from $D$ to $A$, which is the opposite direction of the arc. This does not mean that units are flowing from $D$ to $A$, but that the flow-augmenting route has **reduced** the number of units flowing from $A$ to $D$.

Notice that in updating the numbers along $AD$, you have reduced the number going 'forwards', from $D$ to $A$, by 4 since your flow-augmenting route went from $D$ to $A$. You have similarly increased the number going 'backwards', from $A$ to $D$, in our route by 4.

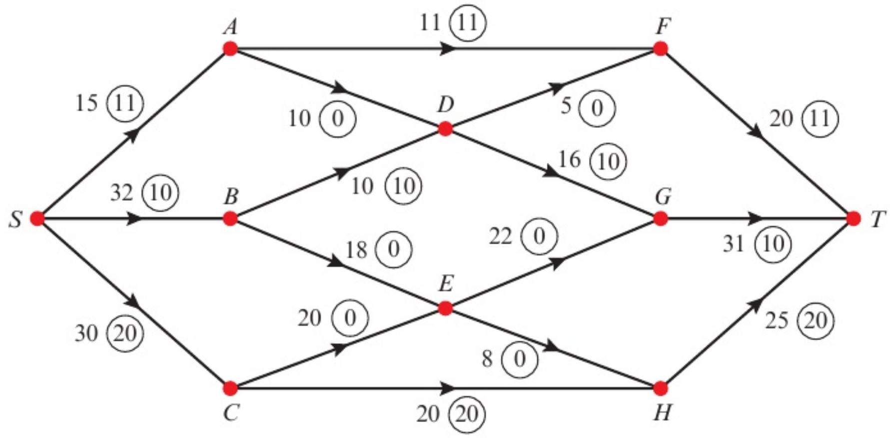
c Start by putting all the arrows on the arcs, using the original diagram. To find the flow on each arc, use the latest numbers from the arrows on each arc which oppose the direction of that arc.




d The value of the maximum flow is 17.

**Note** This is equal to the value of the initial flow plus the value of the flow-augmenting route.

## Example 10




The diagram shows a network of corridors represented by arcs. The corridors are those that will be used in evacuating a large cinema in the event of a fire. The capacity of each corridor is shown on each arc. The numbers in circles represent a possible flow of 41 people per second, from $S$ to $T$.

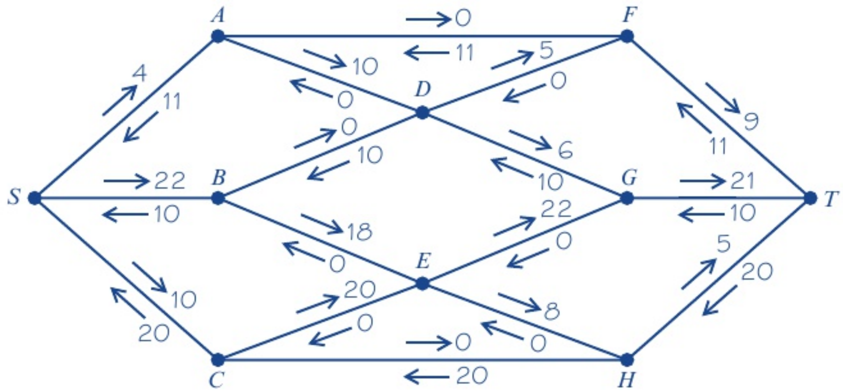
In order for the building to achieve a safety certificate it is necessary to show that it is possible for at least 70 people per second to pass from $S$ to $T$.

**a** Use the initial flow and the labelling procedure to find a maximum flow through the network. You must list each flow-augmenting route you use, together with its flow.

**b** Draw your maximal flow pattern.

**c** Will the cinema achieve its safety certificate?

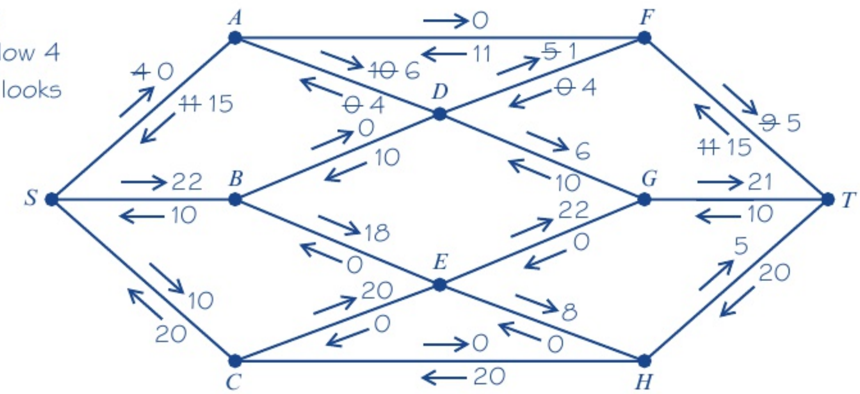
**a** Labelling procedure:




Flow-augmenting routes:

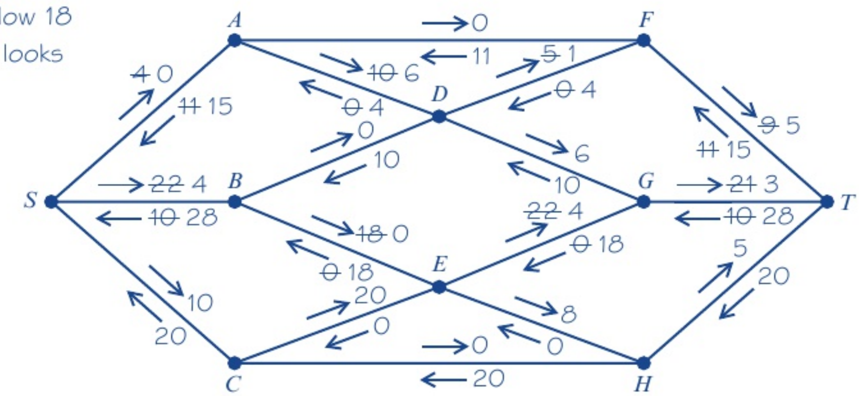
*SADFT* – additional flow 4

The updated diagram looks like this.



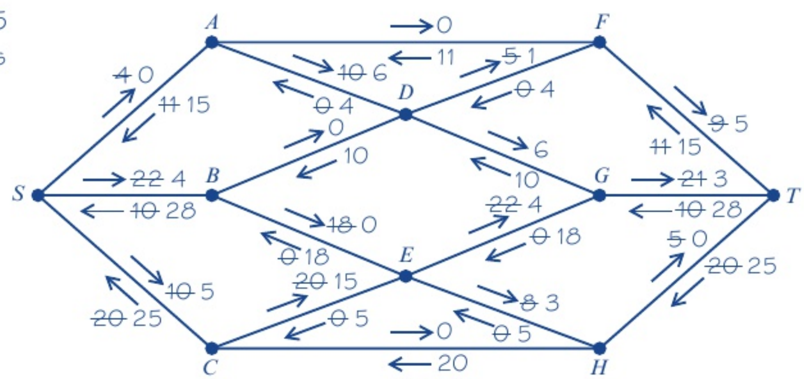

*SBEGT* – additional flow 18

The updated diagram looks like this.



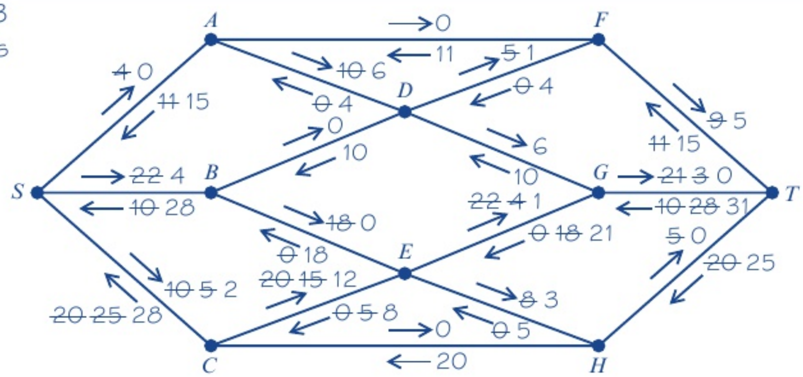

*SCEHT* – additional flow 5

The updated diagram looks like this.



*SCEGT* – additional flow 3

The updated diagram looks like this.




Look for a further route:

It cannot start at *SA* since this arc has zero capacity in the direction *SA*.

It could start at *SB*, but all the exits from *B* have zero capacity in the direction you would need to move, so *SB* is not possible.

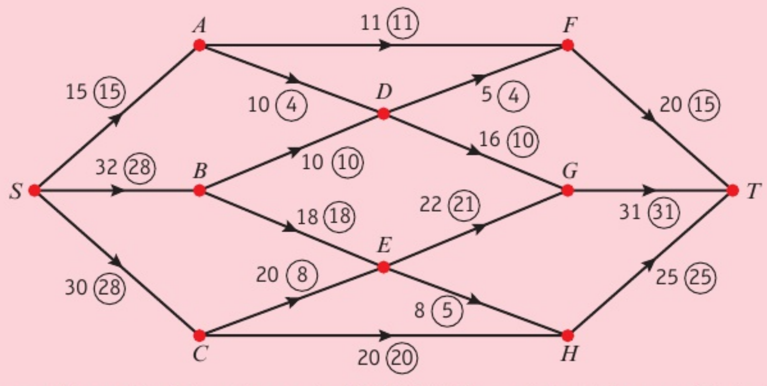
It could start at *SC*, then it must go along *CE*. You cannot go to *H* since *HT* has zero capacity and *HC* takes you back to *C*.

So you must start with *SCEG*. From *G* you cannot proceed to *T*, but you could move to *D*.

So the route starts with *SCEGD* and then proceeds to *F* and to *T*.

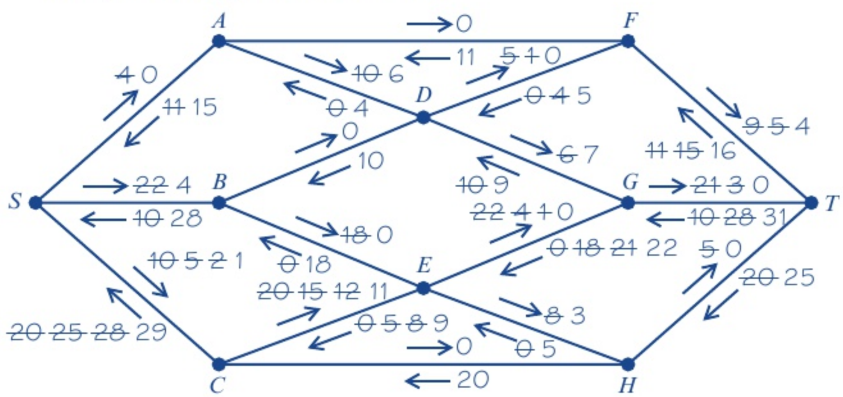
*SCEGDFT* – additional flow 1

**Watch out** This diagram shows the flow represented by the previous stage of the labelling procedure:



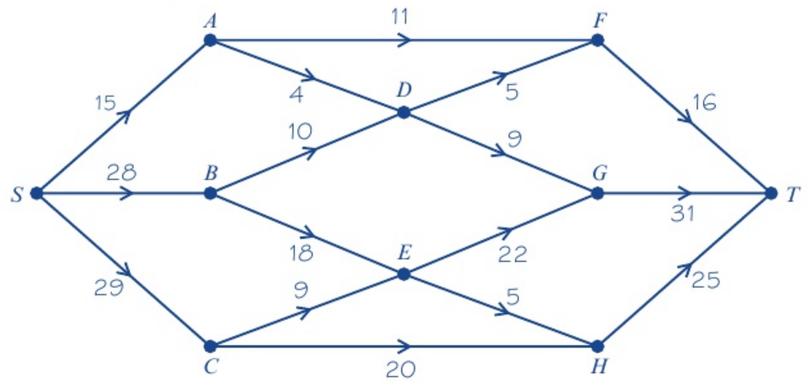

The flow-augmenting route *SCEGDFT* increases the **total flow** in the network by 1. However, if at this stage you wanted to find a route that **one specific** additional person could use to evacuate the cinema, this route would not be suitable, as it uses an arrow from *G* to *D*, whereas this arc flows from *D* to *G*. The additional person would need to follow the route *SCEGT*. This would only be possible by first diverting one person from *DG* to *DF* which creates spare capacity along *GT*.

The updated diagram looks like this.




Notice how the numbers along *DG* have been updated. The number of the forward arrow (from *G* to *D*) has been reduced by 1, and the number on the backward arrow (from *D* to *G*) has been increased by 1.

**b** The final flow pattern is




**c** The final flow pattern above shows a flow of 72. This means the cinema will get its safety certificate, since 72 people per second can leave the cinema.

## Exercise 3D

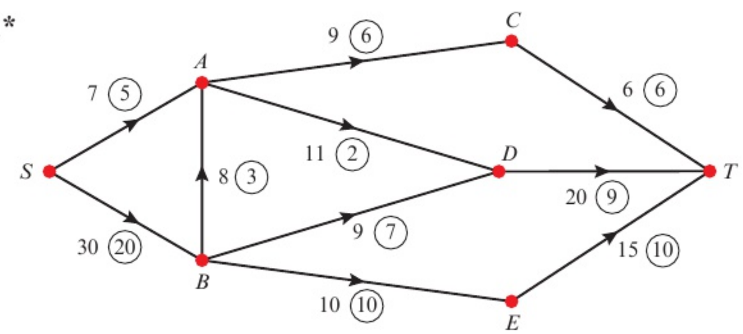
**Answer templates for questions marked * are available at www.pearsonschools.co.uk/d2maths**

**In questions 1 to 4**, the diagrams show capacitated directed networks. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern from $S$ to $T$.

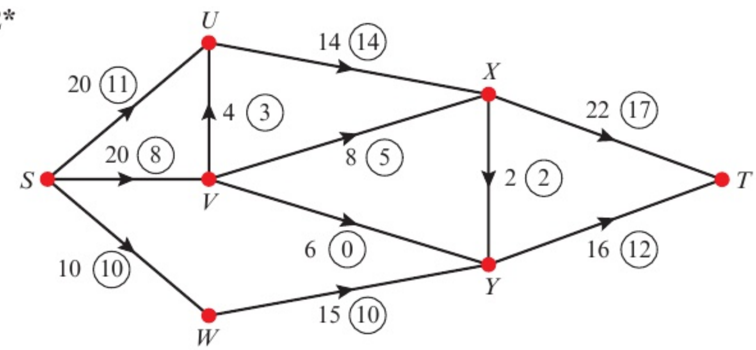
**a** Starting from the initial flow pattern, use the labelling procedure to find a maximum flow through the network. You must list each flow-augmenting route you use together with its flow.

**b** Draw your final flow pattern and state the value of your maximum flow.

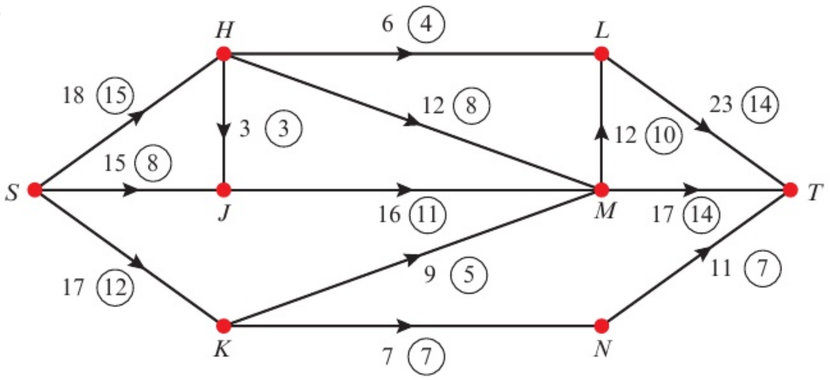
**1***



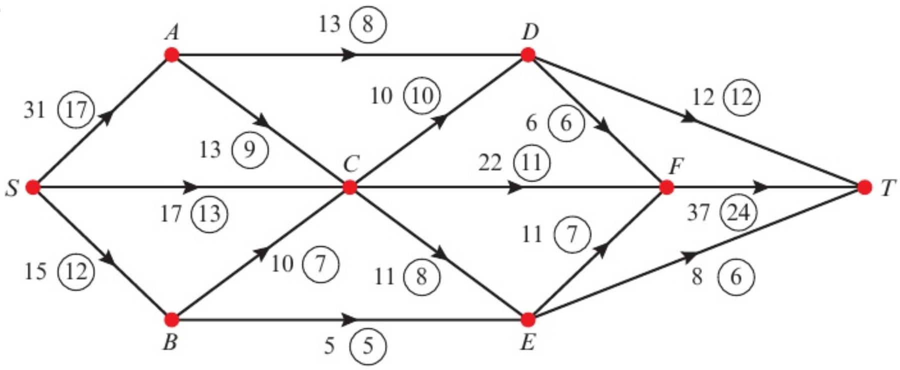

**2***



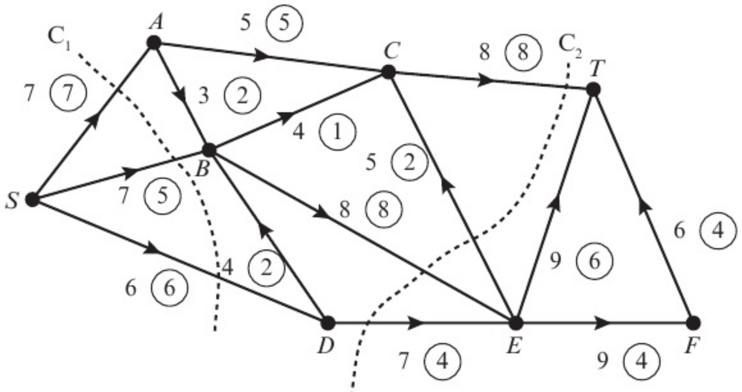
**3***




**4***



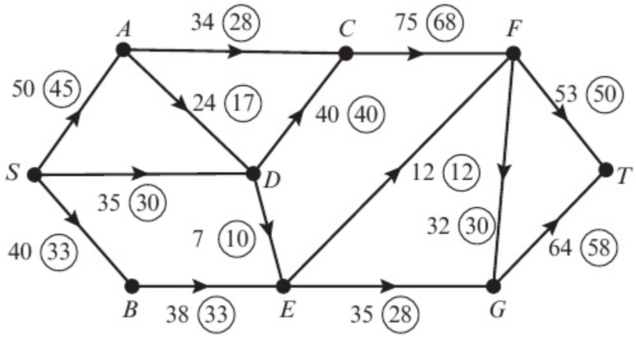

E **5*** The diagram shows a capacitated directed network. The number on each arc represents its capacity and the number in each circle represents an initial flow from $S$ to $T$.




**a** Find the capacity of the two cuts $C_1$ and $C_2$. **(2 marks)**

**b** Use flow augmentation to find the maximum flow.
List each flow-augmenting route, together with its flow. **(4 marks)**

E/P **6*** The diagram shows a capacitated directed network, which is used to model the flow of data in a computer network. The numbers on each arc represent the capacity of the corresponding section of the network and the numbers in circles represent the flow of data in Mb per second.




**a** Find two different flow-augmenting routes which can increase the flow through the network by an extra 5 Mb per second. **(4 marks)**

A single data packet of size 5 Mb is to be sent from $S$ to $T$ over the network.

**b** State, with a reason, which of these two flow-augmenting routes should be used as a path for this data packet. **(1 mark)**

**Problem-solving**

In part **b**, make sure that the flow-augmenting route you select is a valid route for the data packet to take.

## 3.5 Maximum flow–minimum cut theorem

You can prove that a flow is maximal using the maximum flow–minimum cut theorem.

**Notation** This is sometimes abbreviated to **max flow–min cut**.

The difficulty with the labelling procedure is that you may fail to spot a route. Therefore a method of checking whether the improved flow that you have found is maximal is useful.

The maximum flow–minimum cut theorem states:

- **In a network the value of the maximum flow is equal to the value of the minimum cut.**

In passing from source to sink all of the flow must pass over the cut. A useful image is that of a border between two countries, $A$ and $B$, with roads passing from $A$ into $B$. The 'roads' are the arcs and the 'border' the cut. All traffic passing from $A$ to $B$ must pass over the border at some point and it could be used to restrict the number of vehicles entering country $B$. If only 500 vehicles per day are allowed to go from $A$ to $B$, 501 cannot go through.

Thus the minimum cut acts as restriction on the flow.

This means that you have a method of verifying that your flow is maximal.

- **If you can find a cut with capacity equal to your flow, then the flow is maximal.**

To find the minimum cut you need to look for saturated and empty arcs.

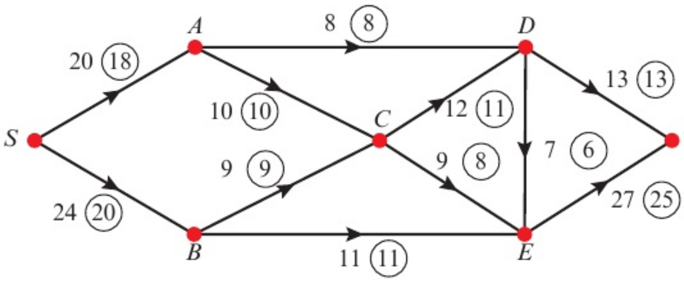
- **The minimum cut passes through:**
  - **saturated arcs if directed from source to sink (into the cut)**
  - **empty arcs if directed from sink to source (out of the cut).**

**Watch out** This does not mean that **every** saturated or empty arc must be in the cut, but it does mean that the cut can **only** pass through saturated arcs directed into the cut or empty arcs directed out of the cut.

## Example 11

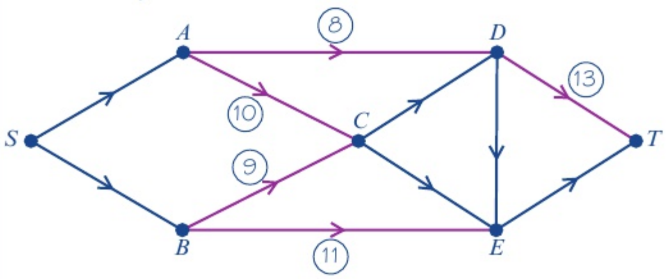
(This is the network used in Example 8.)

The diagram shows a capacitated directed network and a flow of 38 passing through the same network.

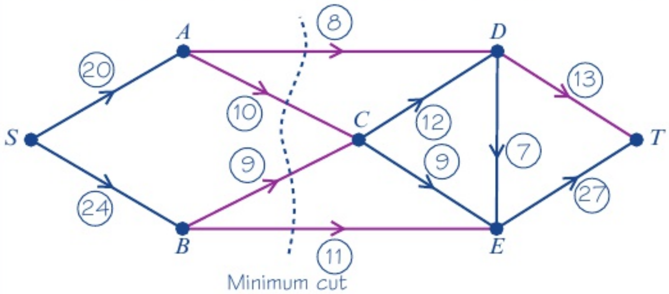



Use the maximum flow–minimum cut theorem to prove that the flow is maximal.

First identify the saturated arcs.




You can draw a cut passing through $AD$, $AC$, $BC$ and $BE$ like this:




The value of our cut is 38, which is equal to the flow given.
Hence, by the maximum flow–minimum cut theorem, the flow is maximal.

Saturated arcs will have a flow equal to their capacity. So you are looking for arcs where the flow matches the capacity.

You have five saturated arcs, $AD$, $DT$, $AC$, $BC$ and $BE$. You do not need to use all of them in the cut, but the cut will pass through some of them.

**Watch out** Make sure you state that you are using the maximum flow–minimum cut theorem.

## Example 12

(This is the network used in Example 9.)

Figure 1 shows a capacitated directed network. Figure 2 shows a flow of 17 passing through the same network.

Use the maximum flow–minimum cut theorem to prove that the flow is maximal.

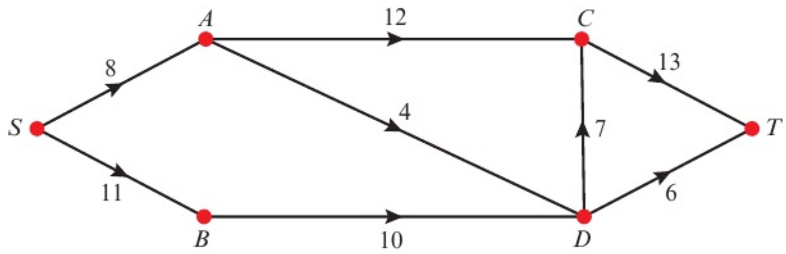



Figure 1

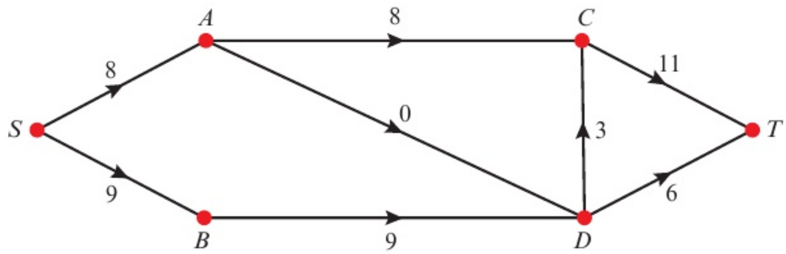
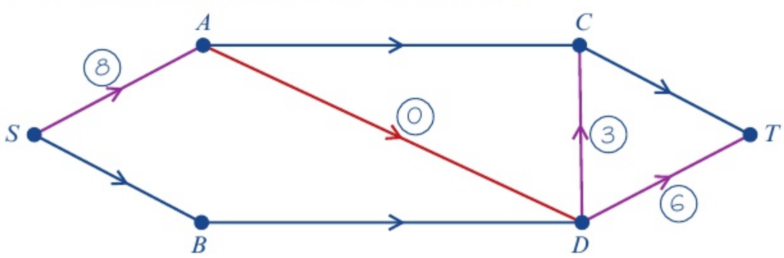
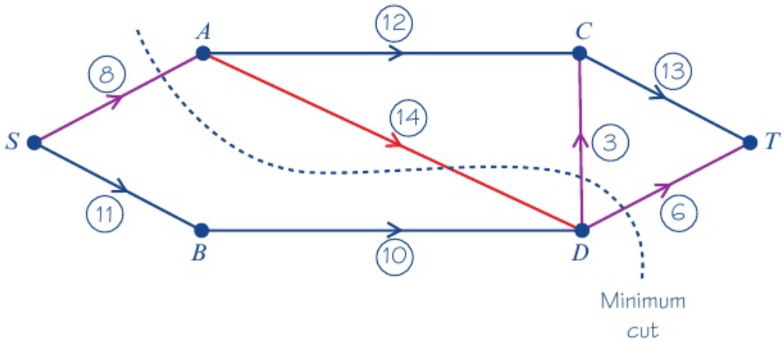



Figure 2

First identify saturated arcs and empty arcs.




Draw a cut passing through $SA$, $AD$, $DC$ and $DT$ like this:




This cut has value 17, which is equal to the flow. Hence by the maximum flow–minimum cut theorem, the flow is maximal.

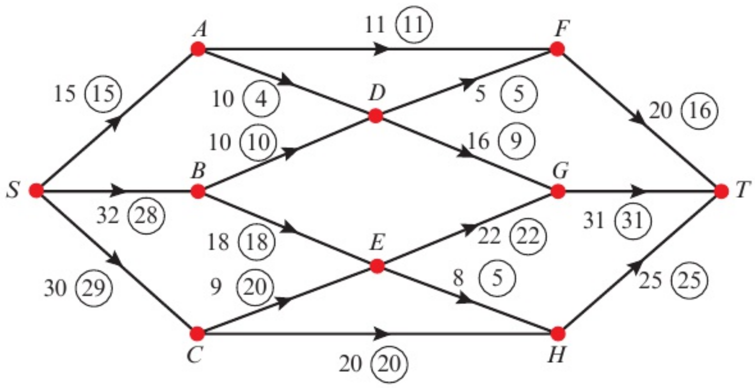
There are only three saturated arcs $SA$, $DC$ and $DT$ and they do not form a cut by themselves. Arc $AD$ is empty. If it were included in the cut through $SA$, $AD$, $DC$ and $DT$ its direction would be 'out' of the cut, and so would contribute zero to the capacity of the cut. This cut passes through three saturated arcs (directed from the set of vertices containing the source to the set of vertices containing the sink) and one empty arc (directed from sink to source), so it must be minimal.

## Example 13

(This is the network used in Example 10)

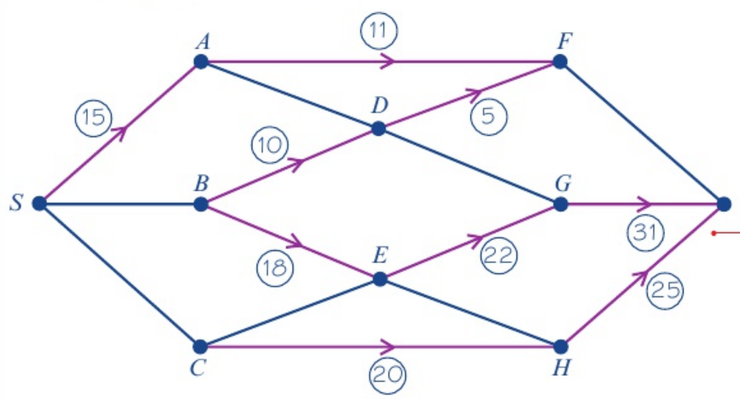
The diagram shows a capacitated directed network and a flow through the same network.

Use the maximum flow–minimum cut theorem to prove that the flow is maximal.

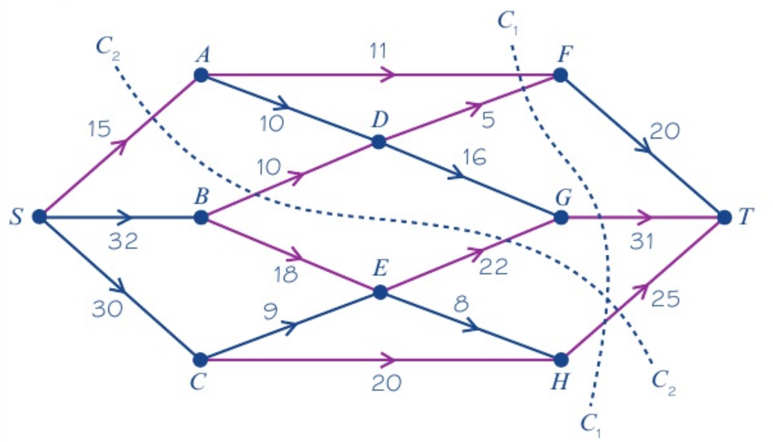



Need to find a cut of capacity 72.

First identify the saturated arcs.




It is possible to find two minimum cuts like this:




Each of these cuts has a value of 72, which is equal to the flow.
Hence, by the maximum flow–minimum cut theorem, the flow is maximal.

Start by working out the value of the flow.

There are nine saturated arcs: $SA$, $AF$, $BD$, $BE$, $CH$, $DF$, $EG$, $GT$ and $HT$.

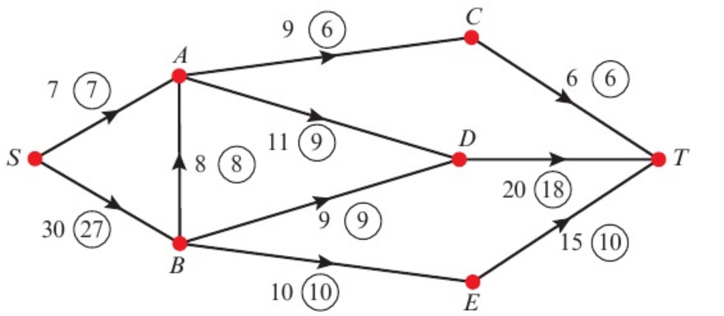
**Watch out** Remember to use the capacity of each arc when calculating the value of a cut. This will only equal the value of the flow if the arc is saturated.

## Exercise 3E

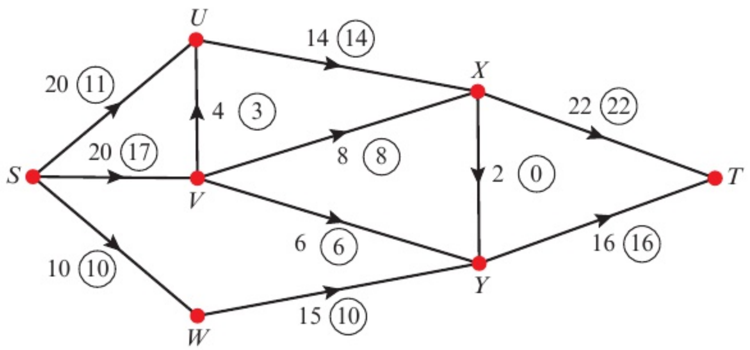
**Answer templates for questions marked * are available at www.pearsonschools.co.uk/d2maths**

Ⓟ **1** Each of the following diagrams shows a capacitated directed network together with a flow. Use the maximum flow – minimum cut theorem to prove that each of these flows is maximal.

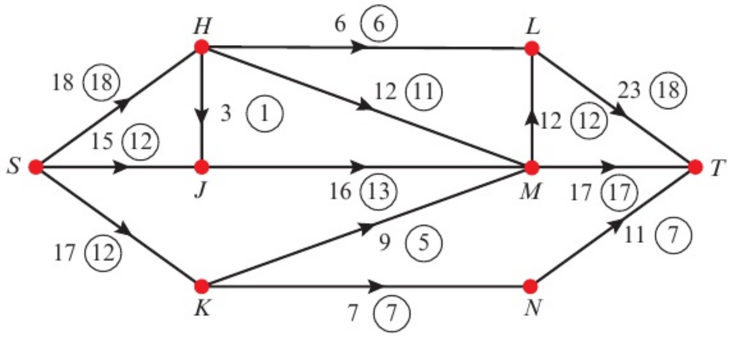
**a**



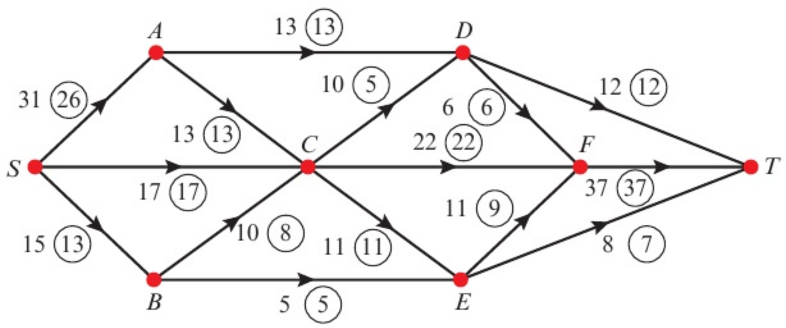

**b**



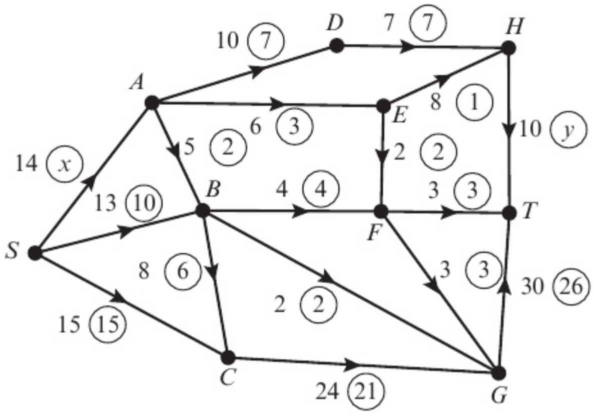

**c**




**d**



E/P **2*** The diagram represents a capacitated directed network. The number on each arc represents the capacity of the arc and the numbers in circles represent initial flows.



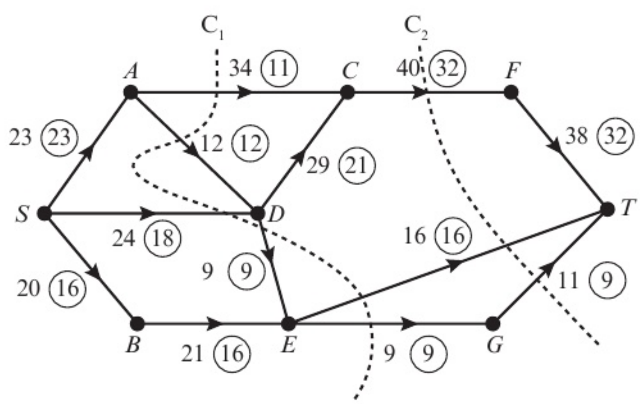

**a** Work out the values of $x$ and $y$. **(2 marks)**

**b** Give the value of the initial flow. **(1 mark)**

**c** Use flow augmentation to find the maximum flow.
List each flow-augmenting route used, together with its flow. **(4 marks)**

**d** By choosing a suitable cut, prove that the flow is maximal.
List the arcs used in your cut. **(4 marks)**

E/P **3*** The diagram shows a capacitated directed network of roads. The number on each arc represents the capacity of the corresponding road in cars per minute. The numbers in circles represent an initial flow.




**a** List the saturated arcs. **(2 marks)**

**b** State the value of the initial flow. **(1 mark)**

**c** State the capacities of the cuts $C_1$ and $C_2$. **(2 marks)**

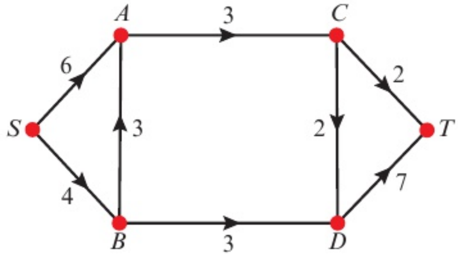
**d** Find, by inspection, a flow-augmenting route that increases the flow by six cars per minute. **(1 mark)**

**e** Prove that the flow is maximal. **(2 marks)**

## Mixed exercise 3

**Answer templates for questions marked * are available at www.pearsonschools.co.uk/d2maths**

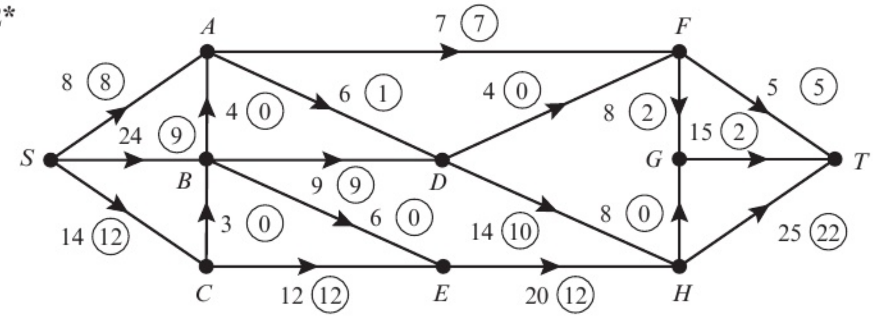
(P) **1*** The diagram shows a capacitated directed network.
The number on each arc indicates the capacity of that arc.




**a** Use the labelling procedure to find the maximum flow through the network from $S$ to $T$, listing each flow-augmenting route you use, together with its flow.

**b** Verify that the flow found in part **a** is maximal.

(E/P) **2***




The diagram shows a capacitated directed network. The number on each arc is the value of the maximum flow along that arc.

**a** Describe briefly a situation for which this type of network could be a suitable model. **(1 mark)**

The numbers in circles show a feasible flow of value 29 from source $S$ to sink $T$.
Take this as the initial flow pattern.

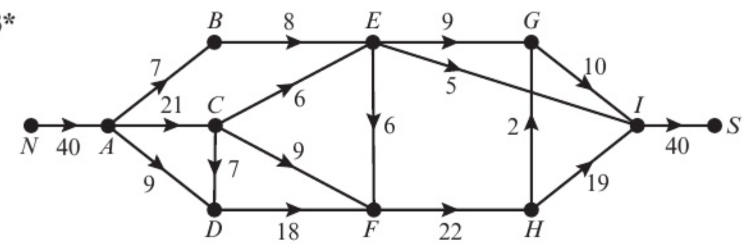
**b** Use the labelling procedure to find the maximum flow through the network from $S$ to $T$. You must list each flow-augmenting route you use together with its flow. **(4 marks)**

**c** Draw your maximum flow pattern and state the final flow. **(2 marks)**

**d** Verify that your answer is a maximum flow by using the maximum flow–minimum cut theorem, listing the arcs through which your cut passes. **(3 marks)**

**e** For the maximum flow, state a property of the arcs found in **d**. **(1 mark)**

(E/P) **3***




The diagram shows the road routes from a bus station, $N$, on the north side of a town to a bus station $S$, on its south side. The number on each arc shows the maximum flow rate, in vehicles per minute, on that route.

**a** State four junctions at which there could be traffic delays, giving a reason for your answer. **(2 marks)**

Given that $AB$, $AD$, $CE$, $CF$ and $EI$ are saturated,

**b** show a flow of 31 from $N$ to $S$ that satisfies this demand. **(2 marks)**

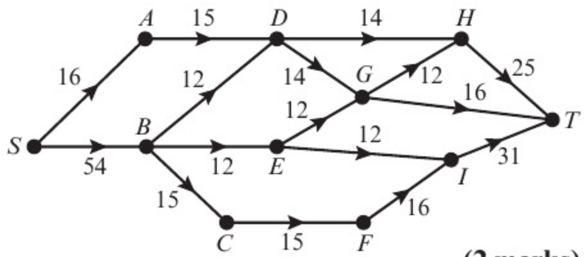
**c** Taking your answer to **b** as the initial flow pattern, use the labelling procedure to find the maximum flow. You should list each flow-augmenting route you use together with its flow. **(4 marks)**

**d** Draw your maximum flow pattern. **(1 mark)**

**e** Verify your solution using the maximum flow–minimum cut theorem, listing the arcs through which your minimum cut passes. **(3 marks)**

**f** Show that, in this case, there is a second minimum cut and list the arcs through which it passes. **(2 marks)**

E/P **4*** The network represents a road system through a town. The number on each arc represents the maximum number of vehicles that can pass along that road every minute, i.e. the capacity of the road.




**a** State the maximum flow along:

**i** $SBCFIT$ **ii** $SADHT$ **(2 marks)**

**b** Show these maximum flows on a diagram. **(1 mark)**

**c** Taking your answer to part **b** as the initial flow pattern, use the labelling procedure to find a maximum flow from $S$ to $T$. List each flow-augmenting route you find, together with its flow. **(4 marks)**

**d** Draw your maximum flow pattern. **(3 marks)**

**e** Prove that your flow is maximal. **(3 marks)**

The council has funding to improve one of the roads to increase the flow from $S$ to $T$. It can choose to increase the flow along one of $BE$, $DH$ or $CF$.

**f** Making your reasoning clear, explain which one of these three roads the council should improve, given that it wishes to maximise the flow through the town. **(2 marks)**

E/P **5*** Figure 1 shows a capacitated directed network. The number on each arc indicates the capacity of that arc.

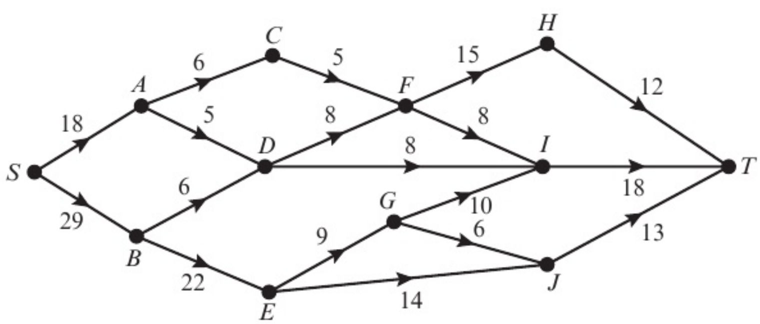



Figure 1

Figure 2 shows a feasible flow of value 29 through the same network.

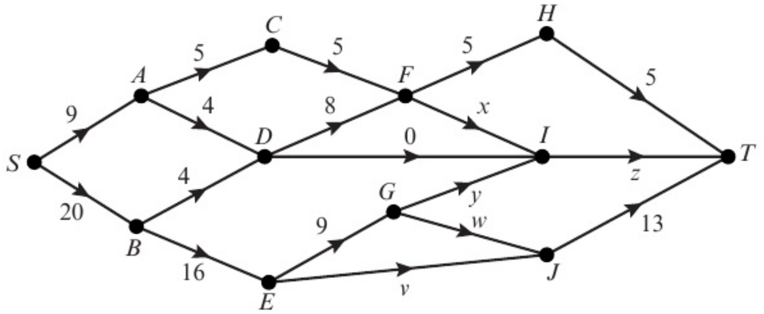



Figure 2

**a** Find the values of the flows $v$, $w$, $x$, $y$ and $z$. **(3 marks)**

Start with the values in Figure 1 and your answers to part **a** as your initial flow pattern.

**b** Use the labelling procedure to find the maximum flow through this network, listing each flow-augmenting route you use together with its flow. **(4 marks)**

**c** Show the maximum flow and state its value. **(1 mark)**

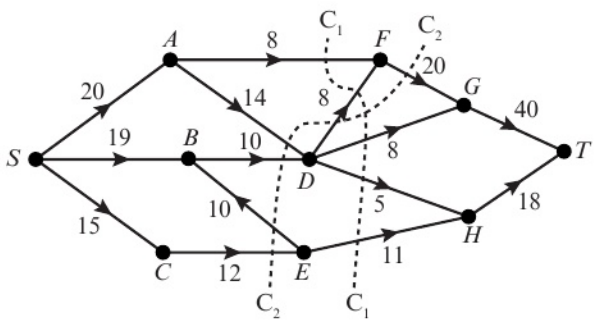
**d** Explain how to find the capacity of a cut in a capacitated directed network. **(2 marks)**

**e** **i** Find the capacity of the cut which passes through the arcs $HT$, $IT$ and $JT$. **(1 mark)**

**ii** Find the minimum cut, listing the arcs through which it passes. **(2 marks)**

**iii** Explain why this proves that the flow in part **c** is a maximum. **(1 mark)**

E/P **6*a** Define what is meant by a cut in a capacitated directed network. **(2 marks)**




The diagram shows a capacitated directed network. The number on each arc indicates the capacity of that arc.

**b** Calculate the values of cuts $C_1$ and $C_2$. **(2 marks)**

Given that one of these cuts is a minimum cut,

**c** state the maximum flow. **(1 mark)**

**d** Deduce the flow along $GT$, making your reasoning clear. **(2 marks)**

**e** By considering the flow into $D$, deduce that there are only two possible integer values for the flow along $SA$. **(3 marks)**

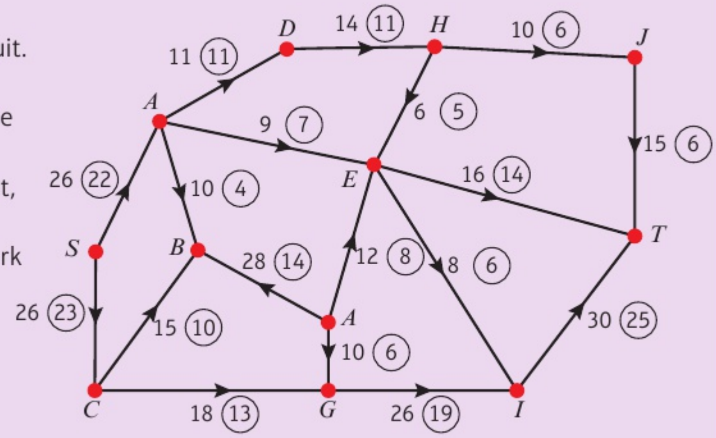
**f** For each of the two values found in part **e**, draw a complete maximum flow pattern. **(4 marks)**

**g** Given that the flow along each arc must be an integer, determine the number of other maximum flow patterns. Give a reason for your answer. **(2 marks)**

## Challenge

The diagram shows part of an electrical circuit. Each arc represents a wire incorporating a diode which only allows current to flow in the direction shown by the arrow. The number on each arc represents the maximum current, in amps, that may flow through the wire. Apart from $S$ and $T$, each node of the network represents an electrical component.

The maximum current flow through any component is 25 amps.




**a** Use flow augmentation to determine the maximum current flow through the circuit.

**b** Explain why the maximum flow–minimum cut theorem cannot be used in this case.

## Summary of key points

**1** A vertex, $S$, is called a **source** if all arcs connected to $S$ are directed away from $S$.
A vertex, $T$, is called a **sink** if all arcs connected to $T$ are directed towards $T$.

**2** To show a **flow** through a network, assign a (non-negative) number to each arc so that it satisfies two conditions:

- the **feasibility condition**, which says that the flow along each arc must not exceed the capacity of that arc.
- the **conservation condition** on all but the source and sink vertices, which says that

  the total flow into a vertex = the total flow out of the vertex

  so the flow cannot 'build up' at a vertex.

**3** If the arc contains a flow equal to its capacity we say that the arc is **saturated**.

**4** The **value** of a flow is the sum of the flows along all arcs leaving a source vertex. This is also equal to the sum of all arcs entering a sink vertex.

**5** A **cut** is a set of arcs whose removal separates the network into two parts, $X$ and $Y$, where $X$ contains at least the source and $Y$ contains at least the sink.

**6** The **capacity (value) of a cut** is the sum of the capacities of those arcs in the cut which are directed from $X$ to $Y$.

**7** When evaluating the capacity of a cut, only include the capacities of the arcs flowing **into** the cut. Arcs which are cut, but whose direction flows out of the cut, contribute zero to the capacity of the cut.

**8** In the labelling procedure, two arrows are drawn on each arc:

- The 'forward' arrow identifies any spare capacity.
- The 'backward' arrow identifies the amount by which the flow in the arc could be reduced.
- Each flow-augmenting route starts at $S$ and finishes at $T$.
- Forward and backward arrows can both be used to find a route, as long as there is capacity in the direction in which you wish to move.

**9** When sending an extra flow of value $f$ along a flow-augmenting route:

- subtract $f$ from the number on each 'forward' arrow on the route,
- add $f$ to the number on each 'backward' arrow on the route.

**10** Sometimes a flow-augmenting route will reduce the flow along a particular arc. This is indicated by using the arrow that points in the opposite direction to the direction of the arc.

**11** The maximum flow – minimum cut theorem states:

In a network the value of the maximum flow is equal to the value of the minimum cut.

**12** If you can find a cut with capacity equal to the flow, then the flow is maximal.

**13** The minimum cut passes through:

- saturated arcs if directed into the cut
- empty arcs if directed out of the cut.

# Flows in networks 2

# 4

## Objectives

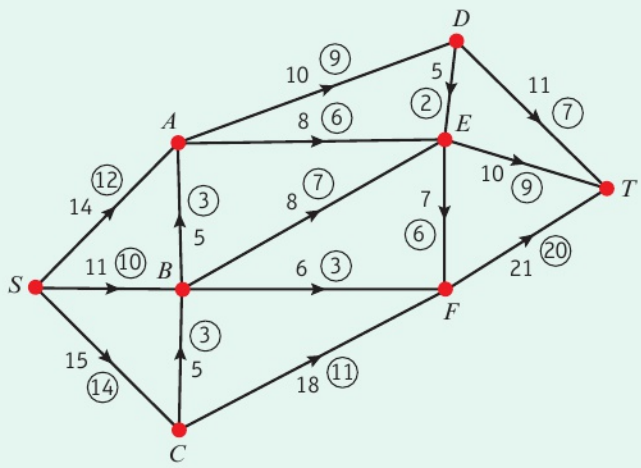
After completing this chapter you should be able to:

- Analyse flows through a network that includes lower capacities → pages 110–119
- Solve problems involving multiple sources and sinks → pages 119–127
- Adapt solutions to deal with nodes of restricted capacity → pages 128–132

In cold weather, it may be necessary to maintain a minimum water flow through pipes to prevent them from freezing. You can model this situation in a capacitated network by assigning lower capacities to arcs. → Exercise 4A, Q2

## Prior knowledge check

**1** The diagram shows a capacitated directed network. The number on each arc represents its capacity and the numbers in circles represent an initial flow.




**a** Find the value of the flow through the network.

**b** Explain what is meant by a saturated arc.

**c** Explain how you know that the given flow is not maximal.

**d** Explain how you know that the value of any cut will be greater than 36. ← Chapter 3

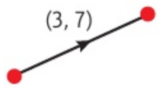
## 4.1 Lower capacities

A

In the previous chapter, each arc of a capacitated directed network had a capacity which represented the maximum possible flow through that arc. This idea will now be extended to situations where, additionally, the flow through an arc cannot fall below some minimum value which is referred to as the **lower capacity** of the arc.

- **In a capacitated directed network, if there are two capacities associated with each arc, the first number represents the lower capacity and the second number represents the upper capacity.**

There are two numbers on this arc. One is the lower capacity of the arc and the other is the upper capacity of the arc.

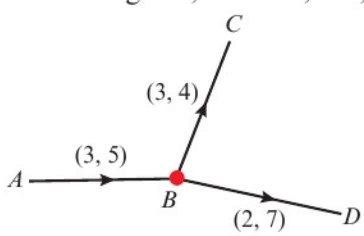



The flow through this arc has a minimum value of 3 and a maximum value of 7.

Dealing with flows in networks that have both upper and lower capacities will require some modification of the notation and procedures developed for networks in which there were only upper capacities.

### Example 1

In the diagram, arcs $AB$, $BC$, and $BD$ represent part of a directed capacitated network.

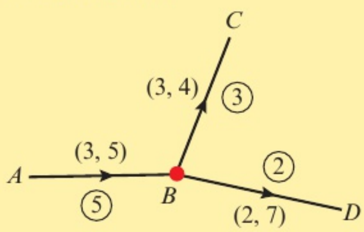



Deduce the value of the flow in each arc.

The minimum flow through $BC$ is 3 and the minimum flow through $BD$ is 2. Thus, the minimum flow out of vertex $B$ is 5.
This requires that the minimum flow into $B$ must be 5.
Since the maximum flow through $AB$ is also 5, the only possible value of the flow through $AB$ is 5. It follows that the flow through $BC$ must be 3 and the flow through $BD$ must be 2.

The **conservation condition** still applies here: total flow into a vertex = total flow out of the vertex.

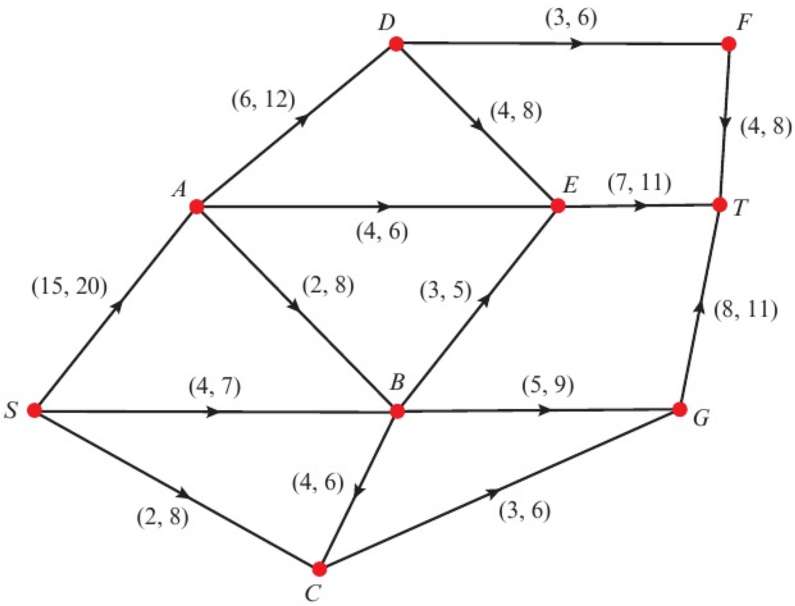
This is the flow:



**A** Under certain conditions, you can use the conservation condition to determine flows on arcs in a network with upper and lower capacities:

- **If the maximum total flow into a vertex is equal to the minimum total flow out of that vertex, then:**
  - **arcs into that vertex are at their upper capacities (saturated)**
  - **arcs out of that vertex are at their lower capacities**
- **If the minimum total flow into a vertex is equal to the maximum total flow out of that vertex, then:**
  - **arcs into that vertex are at their lower capacities**
  - **arcs out of that vertex are at their upper capacities (saturated)**

## Example 2




The diagram shows a directed capacitated network. The numbers on each arc represent the lower and upper capacities of that arc.

**a** By considering vertex $C$, calculate the flow in:

**i** $SC$ **ii** $BC$ **iii** $CG$

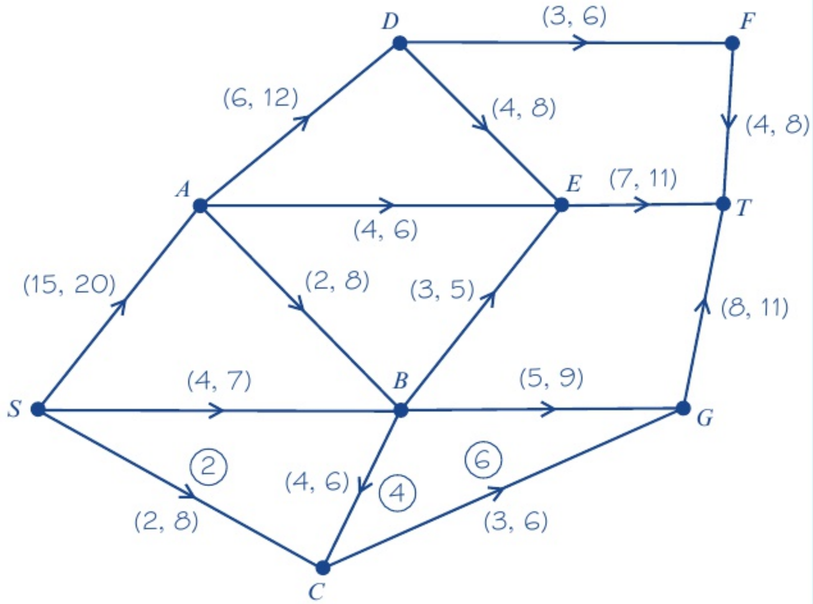
**b** Explain why arcs $DE$, $AE$ and $BE$ must all be at their lower capacities.

**c** Calculate the flow in $GT$.

**A**

**a** The minimum total flow into $C$ is $4 + 2 = 6$ which is the same as the maximum flow out.

The only possibility is that: **i** $SC = 2$ **ii** $BC = 4$ **iii** $CG = 6$

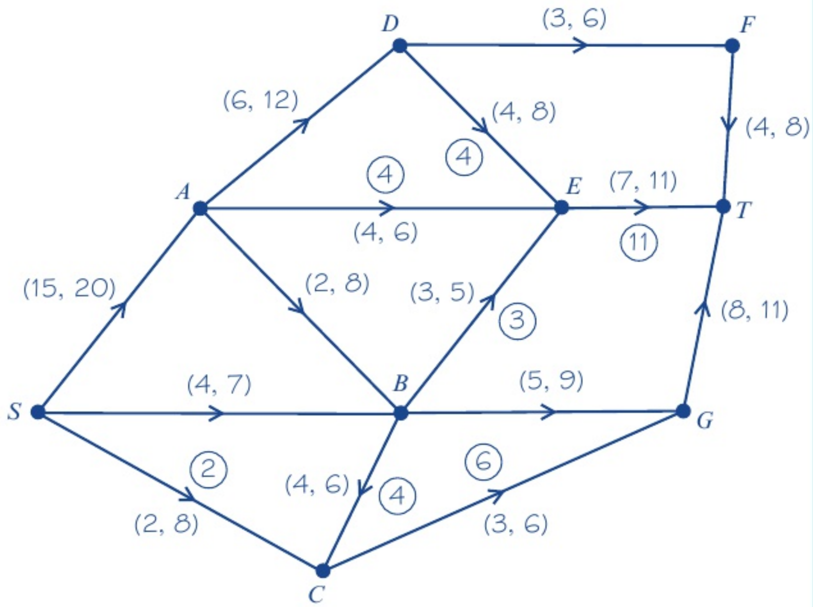



Consider the vertex where these arcs meet. The minimum total flow into this network is equal to the maximum total flow out of it, so arcs $SC$ and $BC$ must be at their lower capacities, and arc $CG$ must be saturated.

**Problem-solving**

Write flows on your diagram as you deduce them.

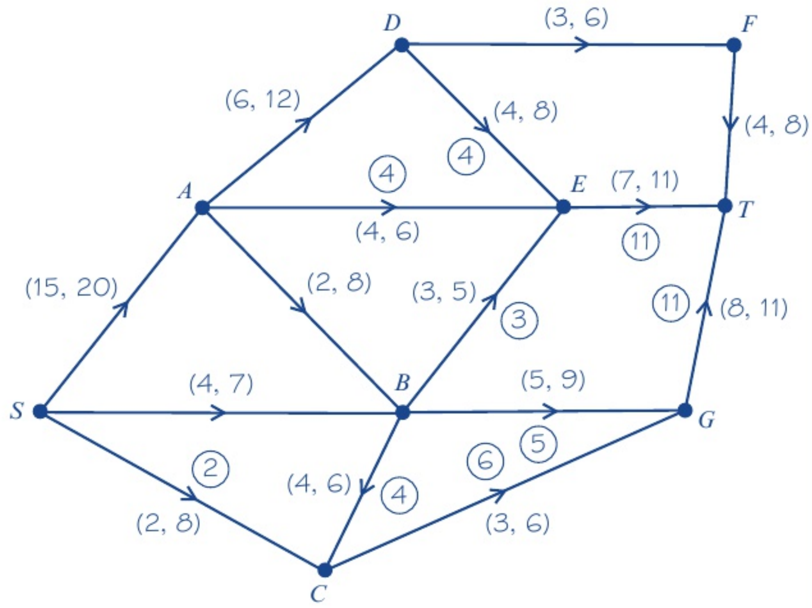
**b** The minimum total flow into $E$ is 11, which is equal to the maximum total flow out. The flow through vertex $E$ must be 11, so $DE$, $AE$ and $BE$ must all be at their lower capacities.




If the flow increased along any one of these arcs, the flow along arc $ET$ would exceed 11, which is not possible.

A

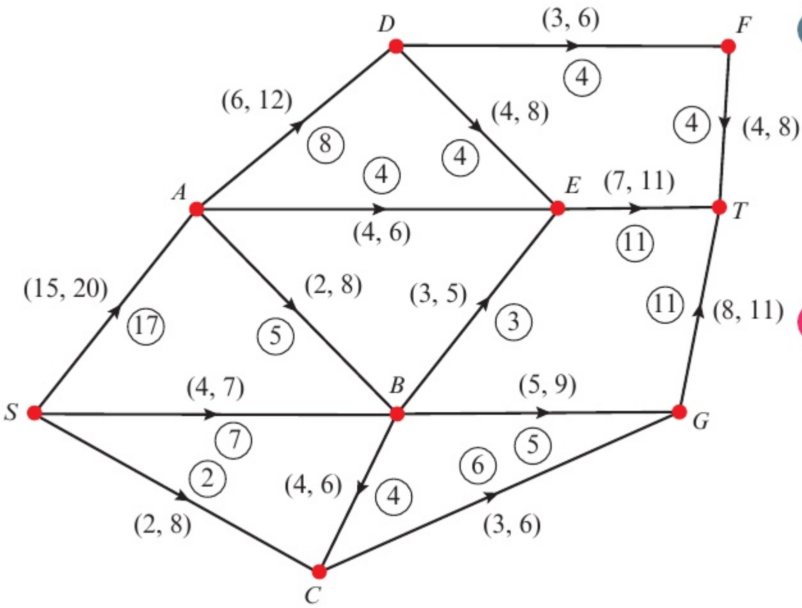
*c* The minimum flow in $BG$ is 5, which, combined with the flow of 6 in $CG$ matches the maximum flow in $GT$. So the flow in $GT$ must be 11.




You already know that the flow along $CG$ must be equal to 6.

## Example 3

The diagram shows a feasible flow for the network given in Example 2.




**Online** Explore feasible flows through a directed network with lower and upper capacities using GeoGebra.

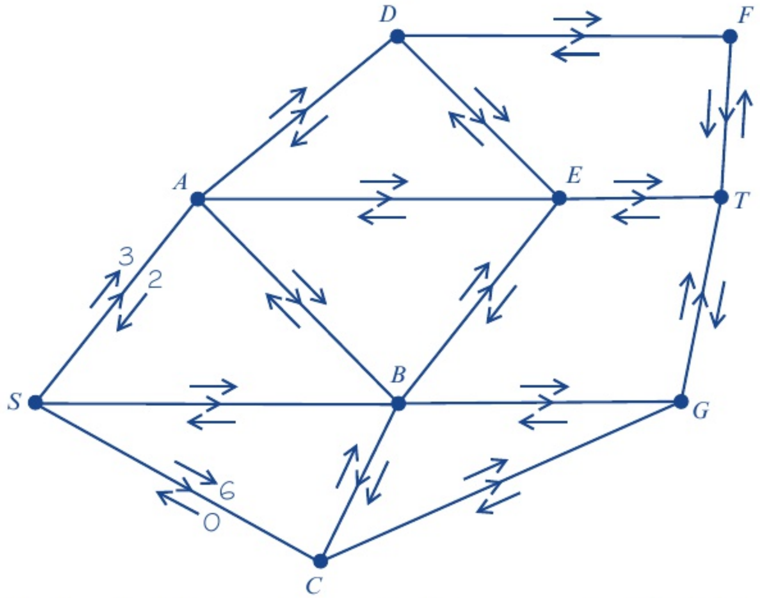
**Watch out** When you are using flow augmentation on a network with lower capacities, you need to be careful when applying values to arrows that oppose the direction of an arc. You can only reduce the flow along any arc to the value of the lower capacity of that arc.

**a** Use flow augmentation to find the maximum flow through the network.

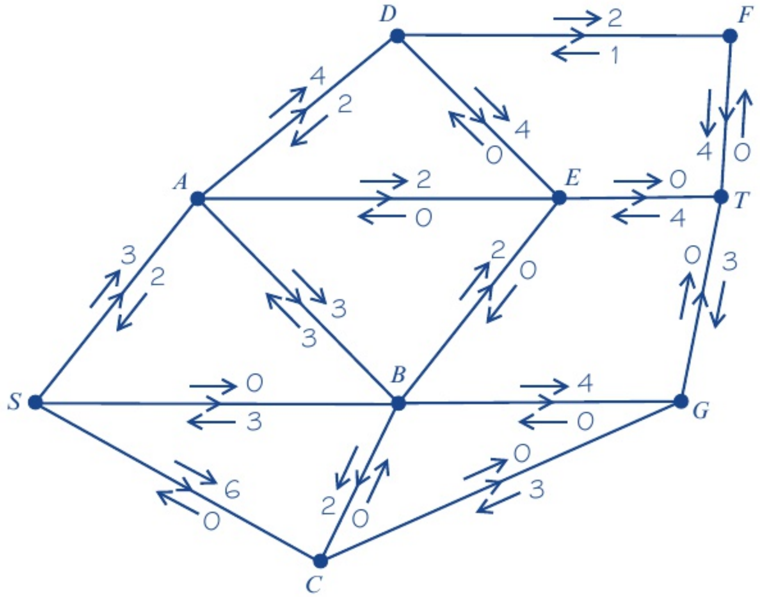
**b** Show the maximum flow on a diagram.

Using the labelling procedure:
The flow along $SA$ can be increased by 3 and can be reduced by 2 to keep within the maximum and minimum capacities.
The flow along $SC$ can be increased by 6, but cannot be reduced as it is already at the minimum level.
This information is shown on the diagram,

**a**



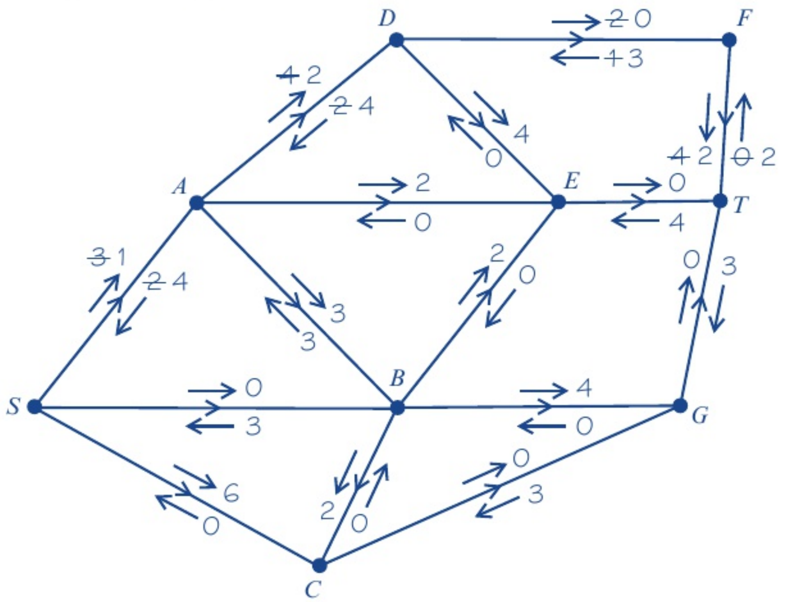

Treating all of the remaining arcs in the same way gives the following diagram.




A flow-augmenting route is $SADFT$ – additional flow 2.

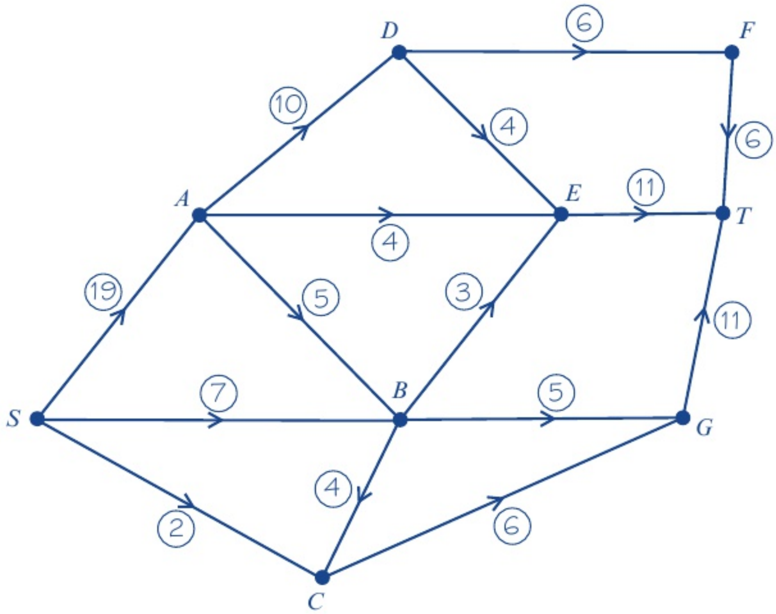
Find a route from the source to the sink which does not use any arrows with 0 values. This will be a flow-augmenting route.

The updated diagram looks like this.



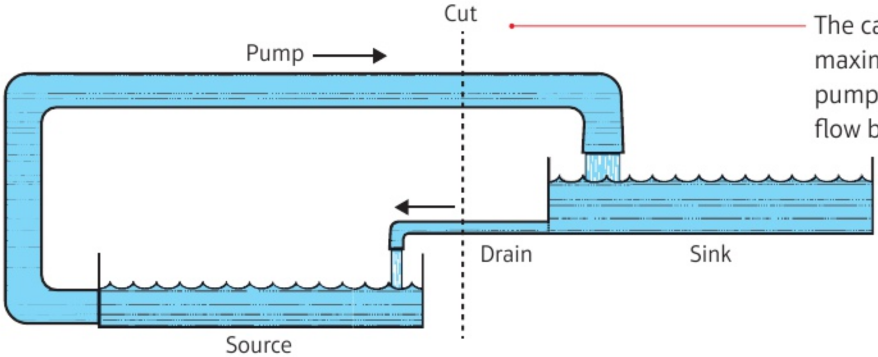

Each of the arcs $DF$, $ET$ and $GT$ is now saturated, so there can be no further augmentation and the flow is maximal.
The initial flow through the network was 17 + 7 + 2 = 26.
Flow augmentation increased this by 2, so the maximum flow is 28.

**b** For $SA$ the flow is given by 15 + 4 = 19
For $AD$ the flow is given by 6 + 4 = 10 and so on.
The flow diagram looks like this.




**Watch out** To find the value of the flow on each arc, look at the arrow that **opposes** the direction of that arc. The value of the flow will be the final value on this arrow **plus** the lower capacity of that arc. For example, arc $SA$ has a lower capacity of 15, so the flow along it is 4 + 15 = 19.

The capacity (or value) of a cut in a network indicates the maximum possible total flow **into** the cut. Any arcs that flow **out** of the cut, and which have lower capacities, will reduce the maximum possible flow into the cut.




The capacity of this cut is equal to the maximum possible flow through the pump **minus** the minimum possible flow back through the drain.

- **If a network includes lower capacities, the capacity (value) of a cut is equal to the sum of the upper capacities crossing the cut from $S$ to $T$, minus the sum of the lower capacities crossing the cut from $T$ to $S$.**

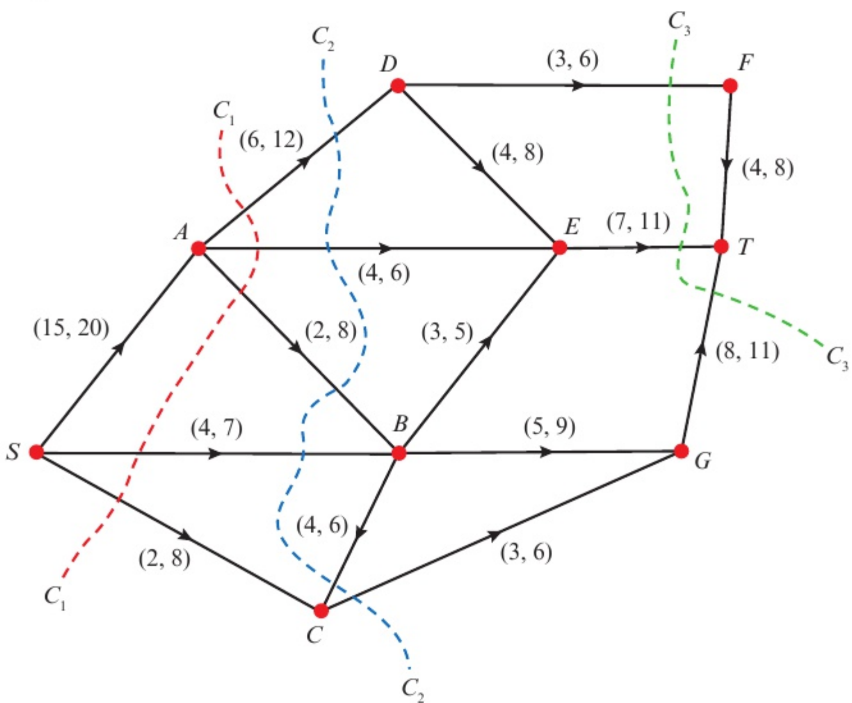
## Example 4

**Online** Explore cuts in a directed network with upper and lower capactities using GeoGebra.

Here is the network used in examples 2 and 3.

**a** Find the capacity of the cuts $C_1$, $C_2$, and $C_3$.

**b** Explain what this shows about the maximal flow.



A

**a** For $C_1$, all of the arcs cross the cut from $S$ to $T$.
The value of the cut is
$12 + 6 + 8 + 7 + 8 = 41$
For $C_2$, the arc $BC$ crosses the cut from $T$ to $S$.
The value of the cut is
$12 + 6 + 8 + 7 - 4 + 6 = 35$
For $C_3$, all of the arcs cross the cut from $S$ to $T$.
The value of the cut is $6 + 11 + 11 = 28$

**b** The value of cut $C_3$ is equal to the feasible flow found in Example 3. By the maximum flow–minimum cut theorem, the flow of 28 is maximal as expected.

Subtract the lower capacity on arc $BC$, which flows out of the cut.

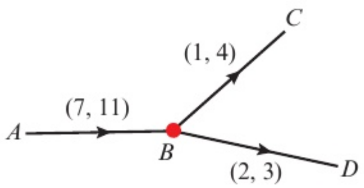
Apply the maximum flow–minimum cut theorem.
← Section 3.5

## Exercise 4A

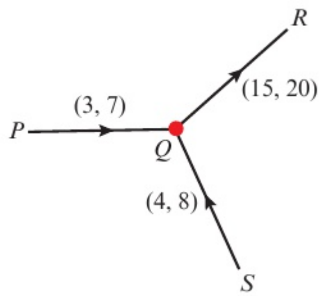
**Answer templates for questions marked * are available at www.pearsonschools.co.uk/d2maths**

**1** Each diagram shows part of a capacitated directed network. The numbers $(x, y)$ on each arc represent the lower and upper capacities on that arc. Deduce the value of the flow in each arc.

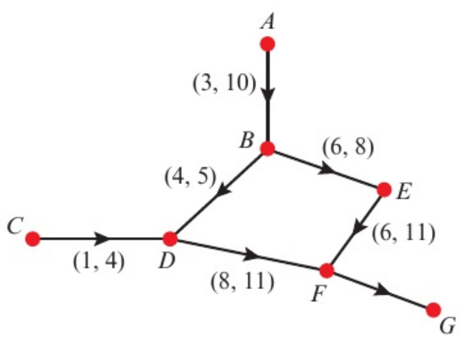
**a**



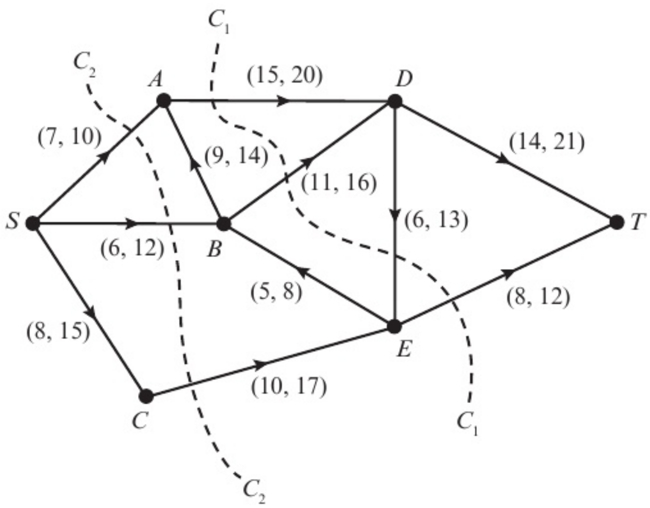

**b**




**c**



A E **2*** The network represents a system of water pipes with lower and upper capacities shown on each arc.

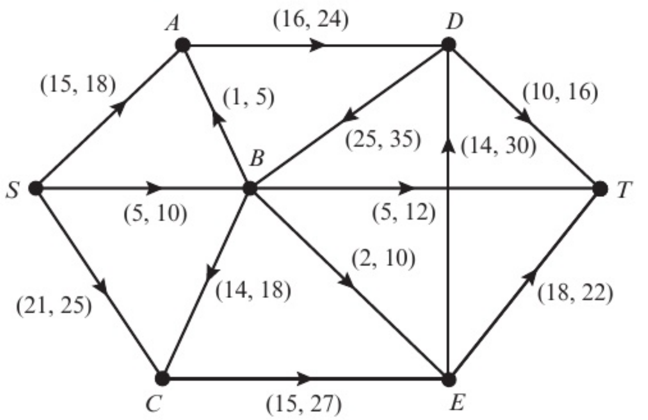



**a** Find the value of each of the cuts $C_1$ and $C_2$. **(3 marks)**

**b** Explain what you can deduce about the maximum flow in the network from your answers to part **a**. **(2 marks)**

**c** Calculate the value of the flow in *SB*, *AB*, *BD* and *BE*. **(2 marks)**

E/P **3*** The diagram represents a capacitated directed network. The numbers on each arc represent the lower and upper capacities of that arc.




**a** Show that there is no feasible flow through the network. **(3 marks)**

A feasible flow can be achieved by increasing the upper capacity of one arc.

**b** **i** Name the arc. **(2 marks)**

**ii** State the minimum required value of its upper capacity. **(1 mark)**

A E/P **4*** Figure 1 shows a capacitated directed network. The numbers on each arc show the lower and upper capacities of that arc.

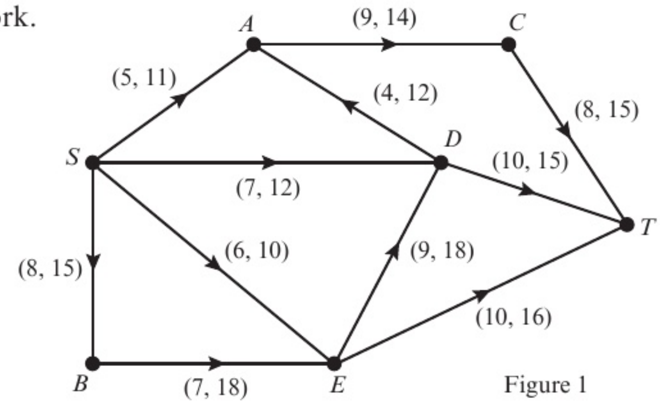


Figure 1

Figure 2 shows a partly completed feasible flow for the network.

**a** Calculate each of the missing values and the flow through the network. **(3 marks)**

**b** Use flow augmentation to maximise the flow through the network. Give each flow-augmenting path and state its value. State the value of the maximum flow. **(4 marks)**

**c** Prove that the flow is maximal. **(2 marks)**

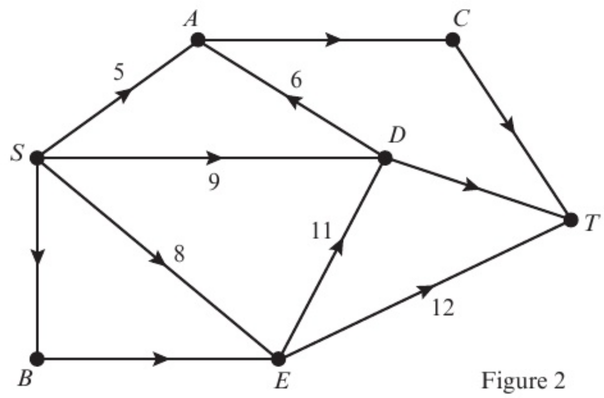
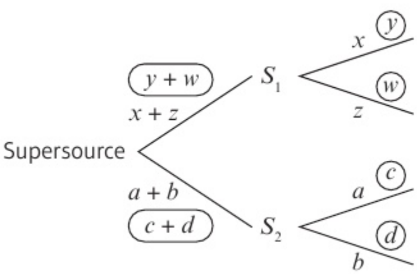


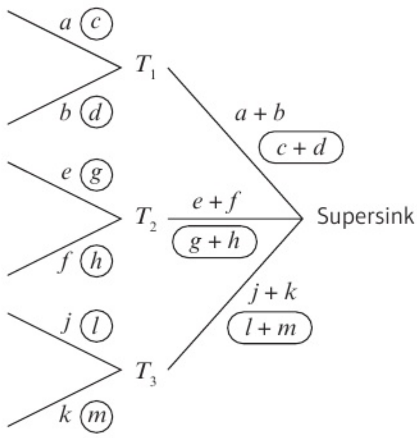
Figure 2

## 4.2 Sources and sinks

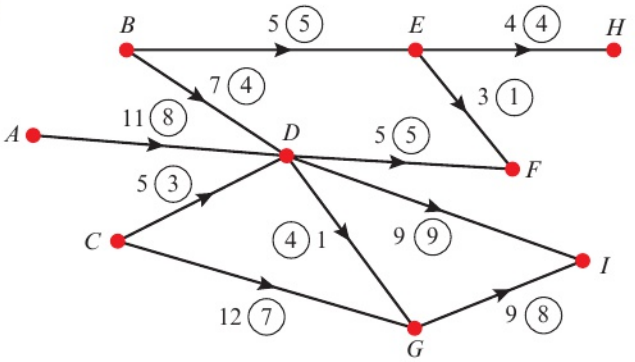
You can deal with networks containing more than one source node and/or more than one sink node by linking each source to a single **supersource**, and each sink to a single **supersink**. You will need to add arcs and choose appropriate capacities and flows for those arcs.

- **The arcs leading from the supersource to each source must have capacity and flow equal to the total capacity and total flow leaving that source.**
- **The arcs leading from each sink to the supersink must have capacity and flow equal to the total capacity and total flow arriving at that sink.**



## Example 5




The diagram shows a capacitated directed network. The capacity of each arc is shown on each arc. The numbers in circles represent an initial flow.

**a** List the source vertices.

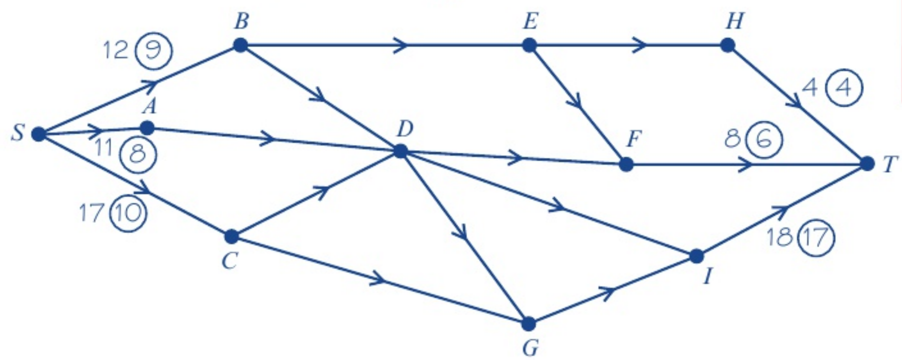
**b** List the sink vertices.

**c** Add supersource, $S$, a supersink, $T$, and appropriate arcs and flows to the diagram.

**a** The source vertices are $A$, $B$, and $C$.

**b** The sink vertices are $F$, $H$, and $I$.

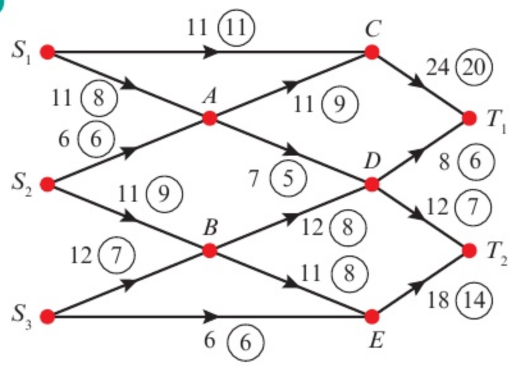
**c**




If all the arcs adjacent to a vertex are directed **away** from that vertex then it is a source. If they are all directed **towards** it, then it is a sink.

Add arcs from the supersource to each source, and from each sink to the supersink.
Look at the capacities and flows out of each of the source nodes, and into each of the sink nodes:
Source $A$: $SA$ must have capacity 11 and initial flow 8.
Source $B$: $SB$ must have capacity 5 + 7 = 12 and initial flow 5 + 4 = 9.
Source $C$: $SC$ must have capacity 5 + 12 = 17 and initial flow 3 + 7 = 10.
Sink $H$: $HT$ must have capacity 4 and initial flow 4.
Sink $F$: $FT$ must have capacity 5 + 3 = 8 and initial flow 1 + 5 = 6.
Sink $I$: $IT$ must have capacity 9 + 9 = 18 and initial flow 9 + 8 = 17.

## Example 6




Water from three reservoirs, $S_1$, $S_2$ and $S_3$ is used to supply two towns $T_1$ and $T_2$, using a network of pipes.

The capacity of each pipe is given by the number on each arc. The numbers in circles represent an initial flow.

**a** Add a supersource, supersink and appropriate arcs to the diagram.

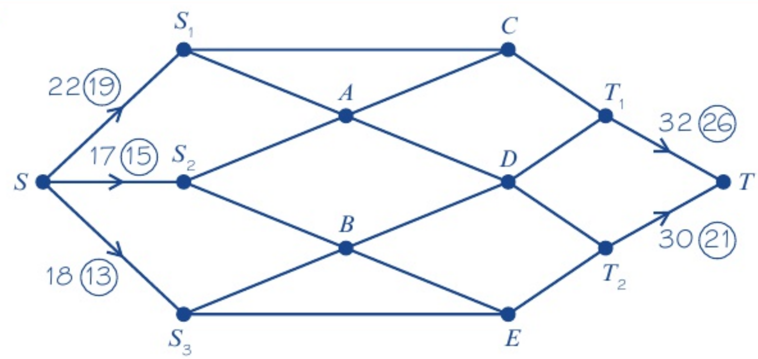
**b** Use the initial flow and the labelling procedure to find the maximum flow through the network. List each flow-augmenting route you use, together with its flow.

**c** Draw your maximal flow pattern.

**d** State the value of your flow.

**e** Show that there are two minimum cuts.

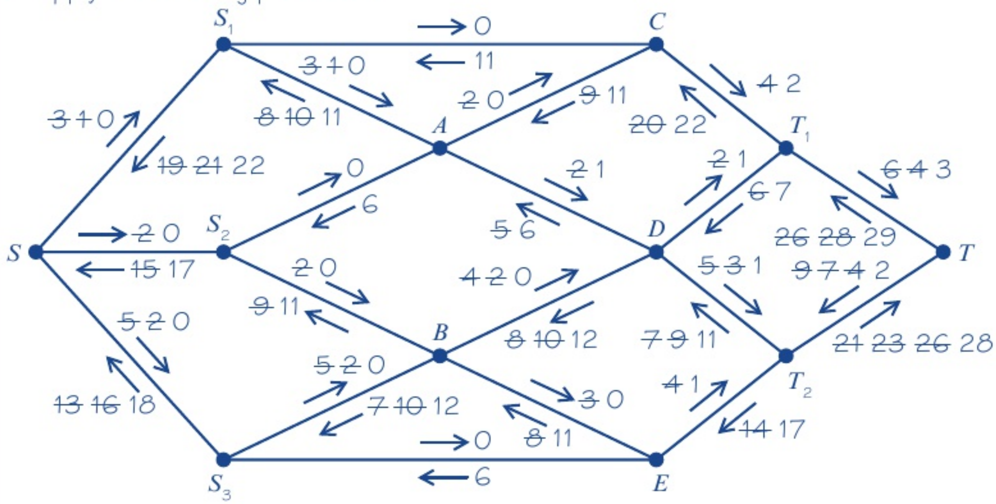
a




Add supersource, $S$, supersink, $T$, and arcs $SS_1$, $SS_2$, $SS_3$, $T_1T$ and $T_2T$ to the network.

Use the total capacities and total flows of the arcs leading out of each source, and into each sink, to determine the capacities and flows on the new arcs. For example, the total capacity out of $S_3$ is 12 + 6 = 18, and the total flow out of $S_3$ is 7 + 6 = 13, so $SS_3$ needs to have capacity 18 and initial flow 13.

**b** Apply the labelling procedure.




A possible set of flow-augmenting routes is:

$SS_1ACT_1T$ – 2

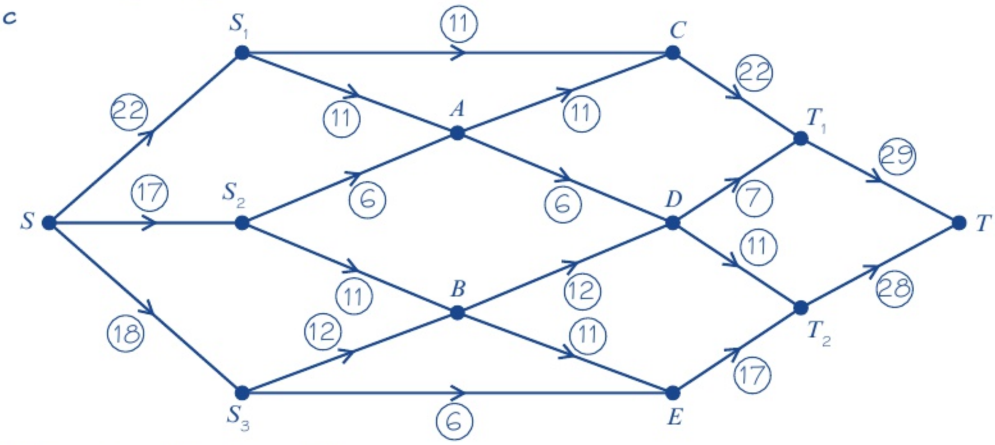
$SS_1ADT_1T$ – 1

$SS_2BDT_2T$ – 2

$SS_3BET_2T$ – 3

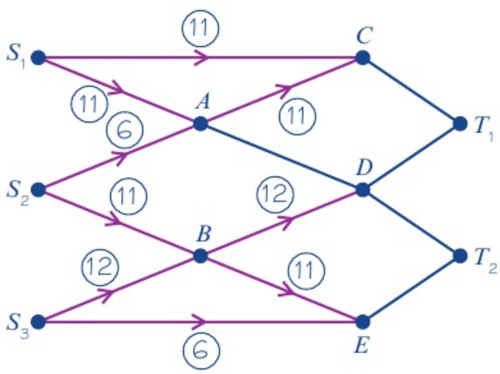
$SS_3BDT_2T$ – 2

**c**




**d** The value of the flow is 57.

**e**

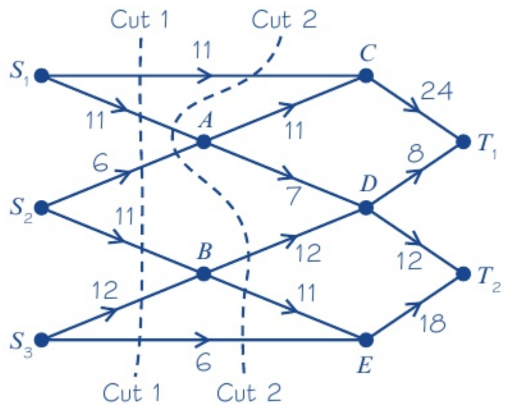



**Watch out** The arcs from the supersource and to the supersink do not correspond to pipes in the network, so they cannot be included in a cut. You need to use the original network diagram when finding cuts.

There are nine saturated arcs, where the current flow is equal to the capacity.

A

You can create two distinct cuts as shown.




Cut 1 = 11 + 11 + 6 + 11 + 12 + 6
= 57

Cut 2 = 11 + 11 + 6 + 12 + 11 + 6
= 57

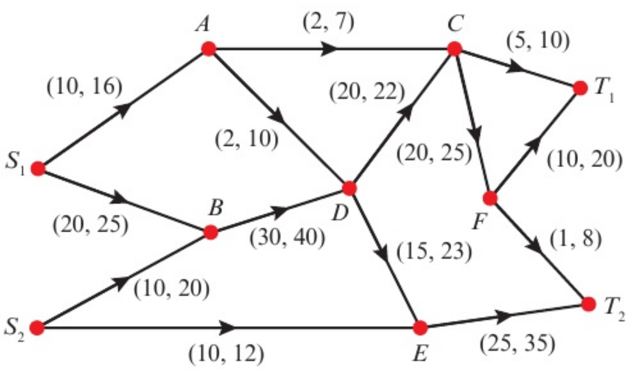
In your exam you may either draw the cuts, or else list all the arcs the cut passes through.

When you are dealing with a network with lower capacities, you need to assign both upper and lower capacities to arcs connected to a supersource or supersink.

- **Each arc from the supersource to a source must have a lower capacity equal to the sum of the lower capacities of the arcs leaving the source and an upper capacity equal to the sum of the upper capacities of the arcs leaving the source.**
- **Each arc from a sink to the supersink must have a lower capacity equal to the sum of the lower capacities of the arcs entering the sink and an upper capacity equal to the sum of the upper capacities of the arcs entering the sink.**

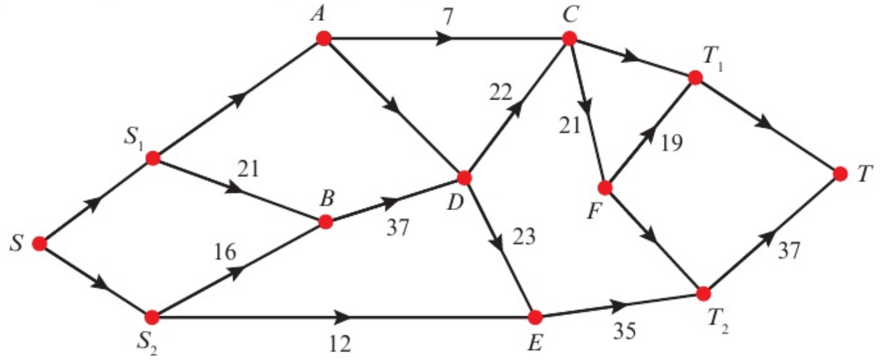
## Example 7

The diagram shows a network of oil pipes with separate sources $S_1$ and $S_2$, and separate sinks $T_1$ and $T_2$. The numbers on the arcs represent the lower and upper capacities through the pipes.




**a** Add a supersource and supersink to the network with appropriate arcs.
Label the new arcs with their lower and upper capacities.

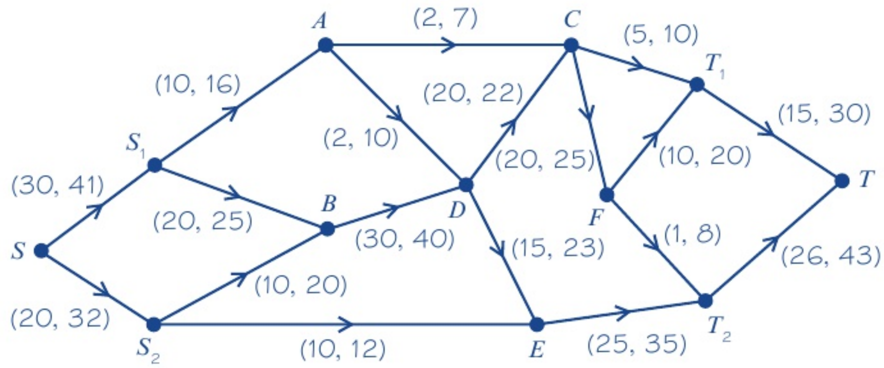
**A** The diagram shows a partly completed feasible flow for the network.




**b** Complete the diagram to show a feasible flow of 64.

**c** Prove that 64 is the maximum flow for the network.

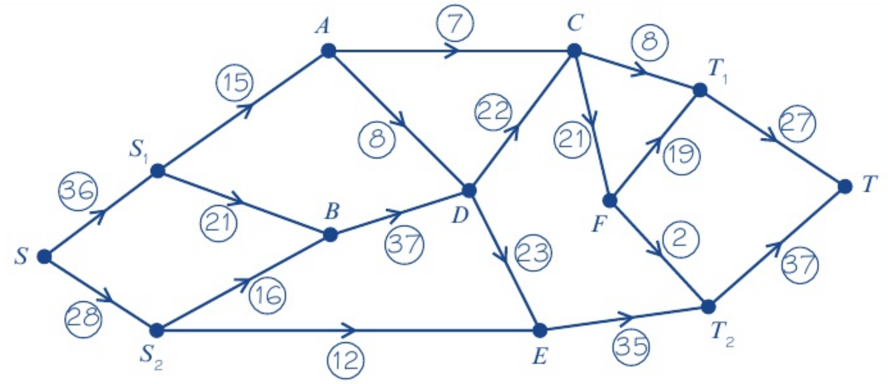
**a**



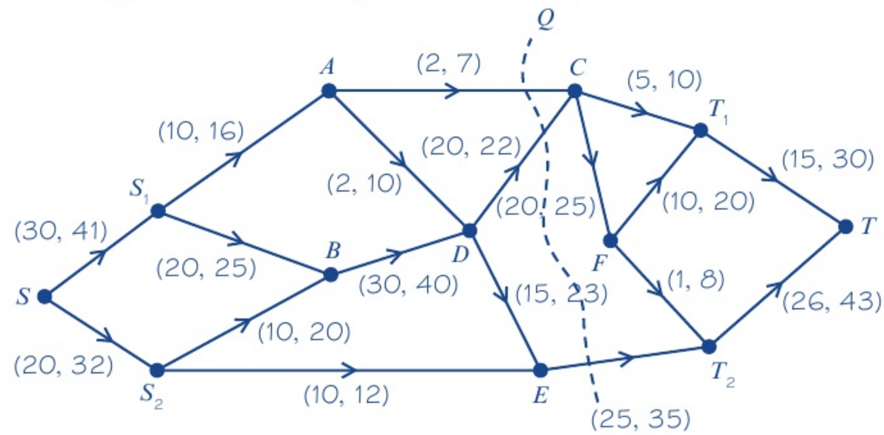

Minimum flow into $S_1$ = total minimum flow out of $S_1 = 10 + 20 = 30$
Maximum flow into $S_1$ = total maximum flow out of $S_1 = 16 + 25 = 41$
Applying the same rules at $S_2$, $T_1$ and $T_2$ completes the diagram.
Note that these rules **only apply** at sources and sinks, they are not generally true at any vertex.

**b** The total flow leaving $S_1$ and $S_2$ is 64, so $S_1A + 21 + 16 + 12 = 64$
This gives $S_1A = 15$
At $S_1$: $SS_1 = 15 + 21 = 36$
At $S_2$: $SS_2 = 16 + 12 = 28$
At $A$: $15 = 7 + AD$, so $AD = 8$
At $C$: $7 + 22 = 21 + CT_1$, so $CT_1 = 8$
At $T_1$: $8 + 19 = T_1T = 27$
At $F$: $21 = 19 + FT_2$, so $FT_2 = 2$
At $T_2$: $35 + 2 = T_2T = 37$

A




**c** The cut, $Q$, shown on the diagram passes through $AC$, $DC$ and $ET_2$.



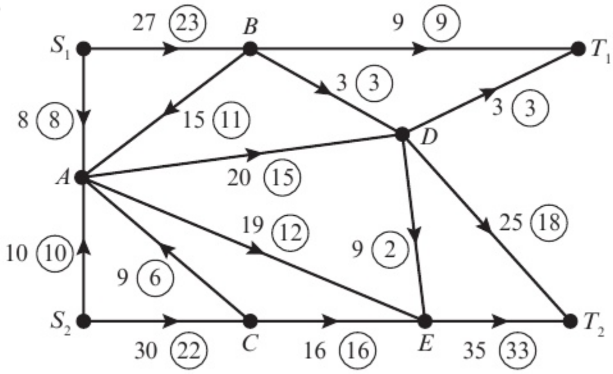

The value of the cut, $Q$, is $7 + 22 + 35 = 64$.
The flow is equal to the value of the cut. Hence, by the maximum flow–minimum cut theorem, the flow must be maximal.

This particular cut was chosen because each of the arcs passed through is saturated, which ensures that the cut has the minimum possible value.

## Exercise 4B

**Answer templates for questions marked * are available at www.pearsonschools.co.uk/d2maths**

E **1***




The diagram above shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow through the network.

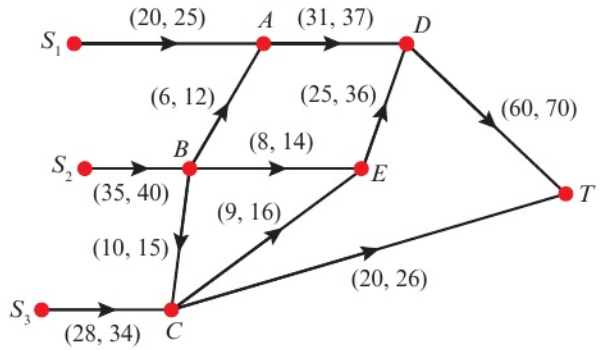
**a** Add a supersource, labelled $S$, and a supersink, labelled $T$, and corresponding arcs to the network. Enter the flow value and maximum capacity of each added arc. **(3 marks)**

A

**b** Use the labelling procedure to find a maximum flow through the network. You must list each flow-augmenting route you use together with its flow, and draw your maximal flow pattern on a copy of the network. **(6 marks)**

**c** Prove that your flow is maximal. **(2 marks)**

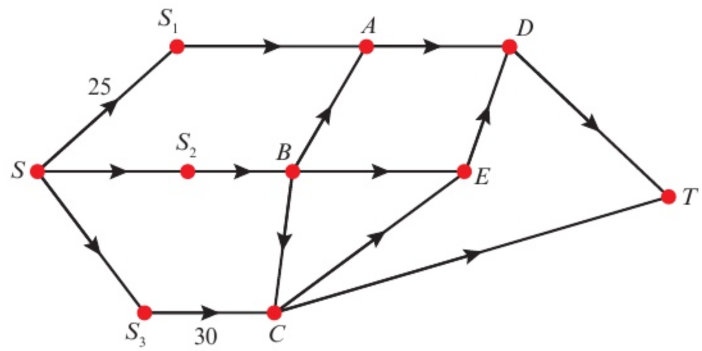
**2*** The diagram represents a system of water pipes showing lower and upper capacities on each arc.



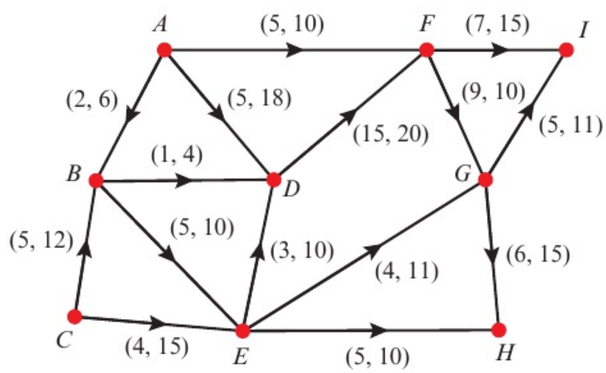

**a** Add a supersource to the diagram with directed arcs showing lower and upper capacities.

**b** Given that arcs *AD*, *BE*, *CE* and *CT* are saturated, find the value of the maximum flow through the network.

**c** Copy and complete this diagram to show the maximum flow.




**3*** The diagram represents a capacitated directed network. The numbers on each arc show the lower and upper capacities of the arc.



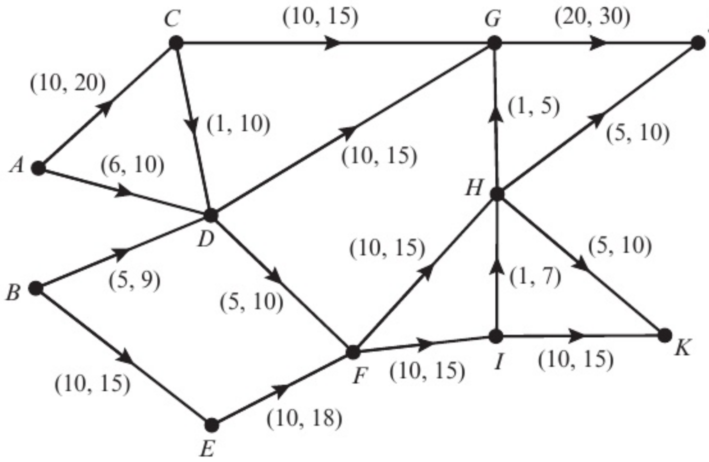

**a** Identify the source vertices and the sink vertices.

A

**b** Add a supersource, $S$, and a supersink, $T$, with connecting arcs to the network labelled appropriately.

**c** Show a flow of 35 through the network.

E/P **4*** The network models the flow of fuel through a system of pipes. The numbers on each arc represent the lower and upper capacities of the corresponding pipe.

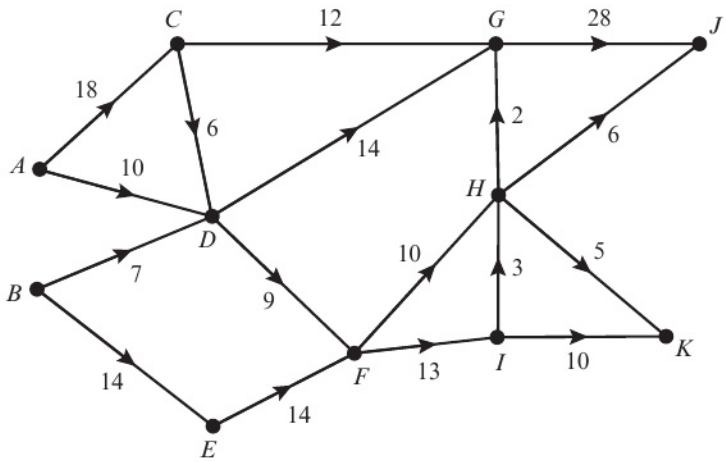



**a** Add a supersource, $S$, and a supersink, $T$, to the network connected by suitably labelled arcs. **(2 marks)**

**b** Find the capacity of the cut that separates $S$, $A$, $B$, $C$, $D$, $E$ and $F$ from $G$, $H$, $I$, $J$, $K$ and $T$. **(2 marks)**

**c** State what can be deduced about the maximum flow through the system from your answer to part **b**. **(2 marks)**

**d** The diagram shows a flow of 49 through the system. Find the maximum flow. You must list all flow-augmenting routes you use, together with their flow. **(3 marks)**




**e** Prove that the flow found in part **d** is maximal. **(3 marks)**

## 4.3 Restricted capacity nodes

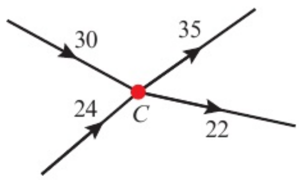
A In all of the models considered so far, flow is only considered to be restricted along the **arcs** in a network. It is assumed that there are no upper or lower restrictions on the flow through any given **vertex**.

In reality, a vertex might represent a network router, road junction or pipe connection with a limited capacity.

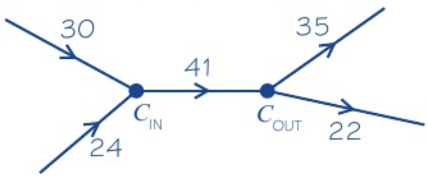
- **If a node (vertex) has restricted capacity then, on the network diagram, it may be replaced by two nodes joined by an arc of that capacity. Flow problems for the network can then be solved in the usual way.**

### Example 8

The diagram shows part of a capacitated directed network.
The node, $C$, has a restricted capacity of 41.
Show this information on the diagram.




The diagram may be modified to represent the restricted capacity at $C$.




**Problem-solving**

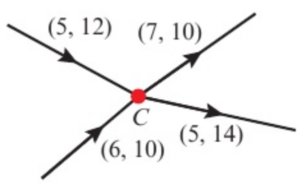
The node $C$ has been replaced with $C_{IN}$ and $C_{OUT}$ joined by an arc of capacity 41. This is an equivalent system, and may now be treated in the usual way when calculating possible flows.

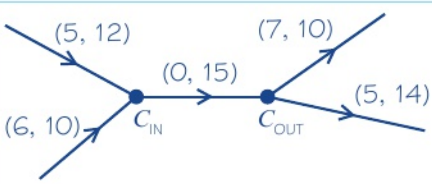
Flows into $C$ are directed towards $C_{IN}$ and flows out of $C$ are directed from $C_{OUT}$. The new arc of capacity 41 is directed from $C_{IN}$ to $C_{OUT}$.

The method can also be adapted to situations where the arcs have upper and lower capacities.

### Example 9

The diagram shows part of a capacitated directed network.
The node, $C$, has a restricted capacity of 15.
Show this information on the diagram.



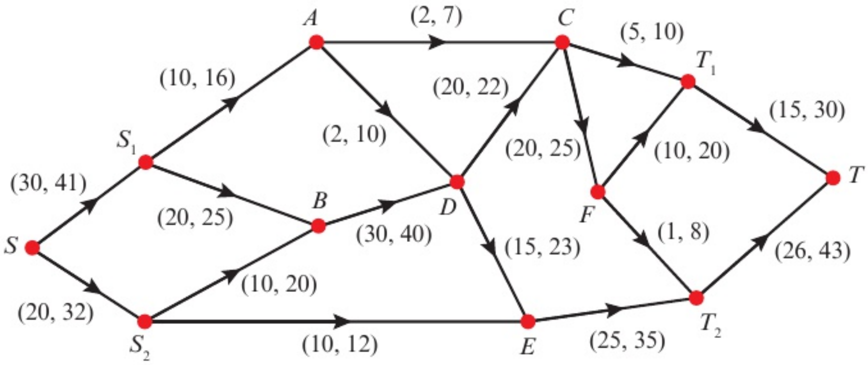




No minimum capacity is given, so this may be set as 0.

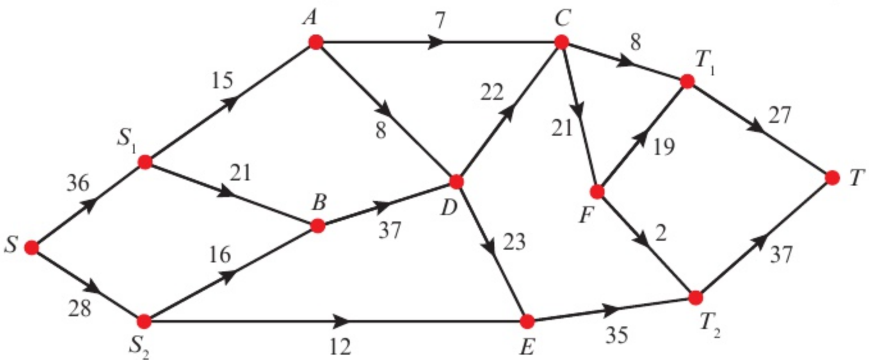
## Example 10

A

Here is the diagram used in Example 7.



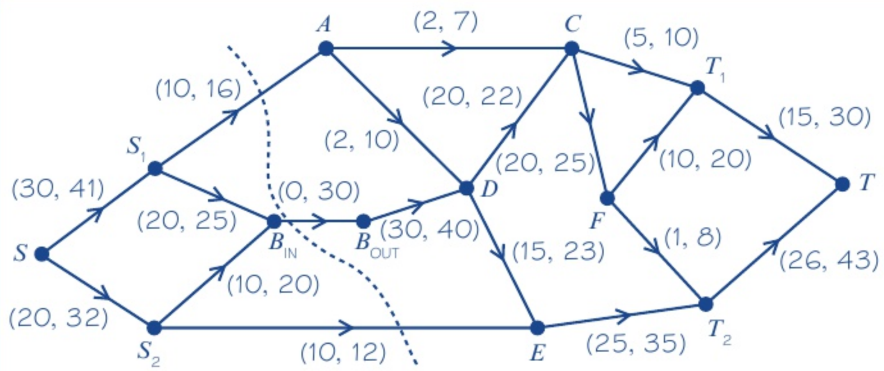

The maximum flow through the network is 64, as shown in this diagram.




The node at $B$ now has a restricted capacity of 30.

**a** Explain why the maximum flow in the network cannot now exceed 58.

**b** Show a feasible flow of 58 for the new network.

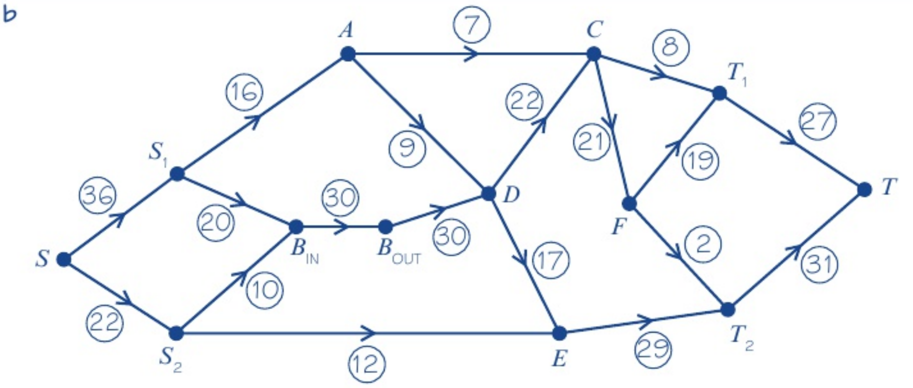



**a** $S_1A$, $B_{IN}B_{OUT}$, $S_2E$ is a cut of capacity 16 + 30 + 12 = 58. Hence by the maximum flow–minimum cut theorem, the total flow in the network cannot exceed 58.

You are allowed to include an arc representing a node restriction in a cut.

A

b




Now the flow through $BC$ has been reduced to 30, so the flows through $S_1B$ and/or $S_2B$ must be reduced. Since $S_2E$ is already saturated, reduce $S_1B$ as much as possible first, and divert additional flow through $S_1A$.
The flow through $DC$ and/or $DE$ must also be reduced, so the additional flow through $S_1A$ can now be routed via $D$.

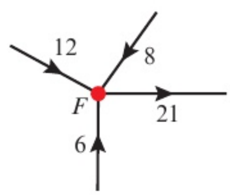
In some cases a node may be completely blocked. You could deal with this in the same way by replacing it with two nodes connected by an arc of capacity 0. However, it is simpler in this case to just delete the node.

- **In the special case where a node is completely blocked, the node and any arcs linked directly to it may be removed from the network.**

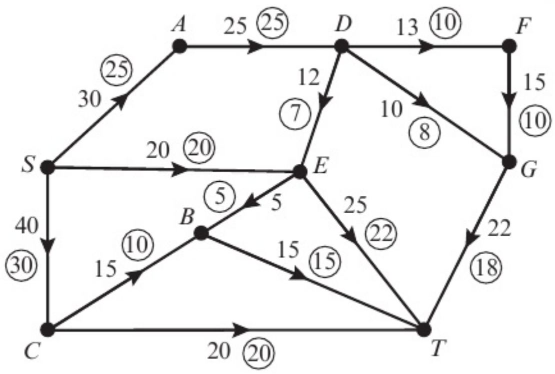
## Exercise 4C

**Answer templates for questions marked * are available at www.pearsonschools.co.uk/d2maths**

**1** The diagram shows part of a capacitated directed network.
The node, $F$, has a restricted capacity of 24.
Show this information on the diagram.




E/P **2*** The diagram represents a system of water pipes. The number on each pipe represents its capacity in litres/second and the numbers in circles represent the current values of the flow in each pipe.




**a** Find the value of the flow through the system and show that it is maximal. **(3 marks)**

A The vertex at $B$ becomes blocked.

**b** Draw a diagram showing the new maximal flow pattern. **(3 marks)**

**c** State the value of the flow and prove that it is maximal. **(3 marks)**

E/P **3*** Figure 1 shows a capacitated directed network. The numbers on each arc show the lower and upper capacities of the arc.

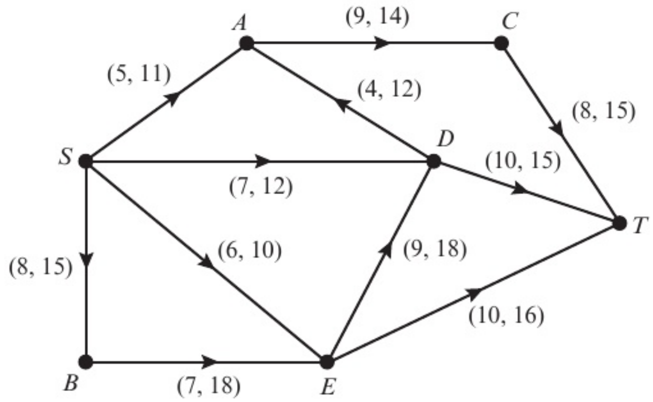



Figure 1

The maximum flow through the network is 45, as shown in Figure 2.

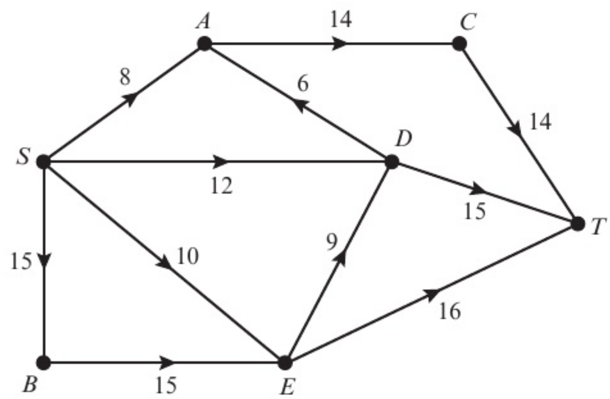



Figure 2

The flow through $D$ is now restricted to 16.

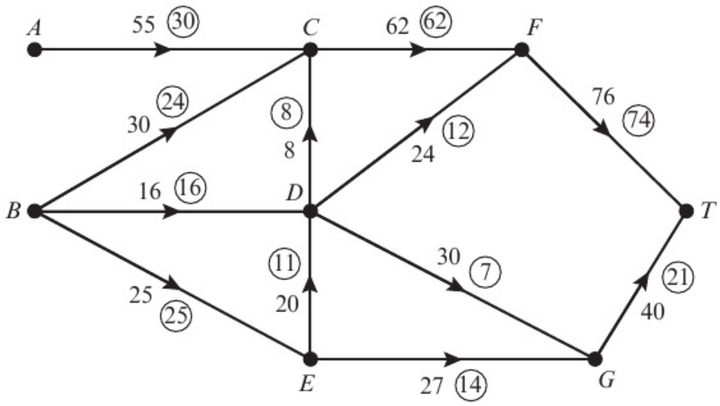
**a** Show this information on a diagram. **(1 mark)**

**b** Show a flow pattern for a feasible flow of 40. **(2 marks)**

**c** Use flow augmentation to find the new maximum flow. State any flow-augmenting path used, together with its value. **(3 marks)**

**d** Find a cut with the same capacity as the maximum flow found in part **c**. **(2 marks)**

A E/P **4** The diagram represents the flow of people through a system of one-way corridors in a school. Each arc represents a corridor and the number on the arc represents the number of students who can travel down that corridor per minute. Each vertex represents a junction where corridors meet. The circled numbers represent a particular flow of students along the network.




**a** Add a supersource, $S$, to the network with appropriately labelled arcs. **(1 mark)**

**b** Find the value of the maximum flow through the system. **(3 marks)**

**c** Prove that the flow is maximal. **(3 marks)**

The school puts up a display of student sculpture at junction $C$, limiting the flow through that junction to 50 students per minute.

**d** Show this restriction on the diagram. **(1 mark)**

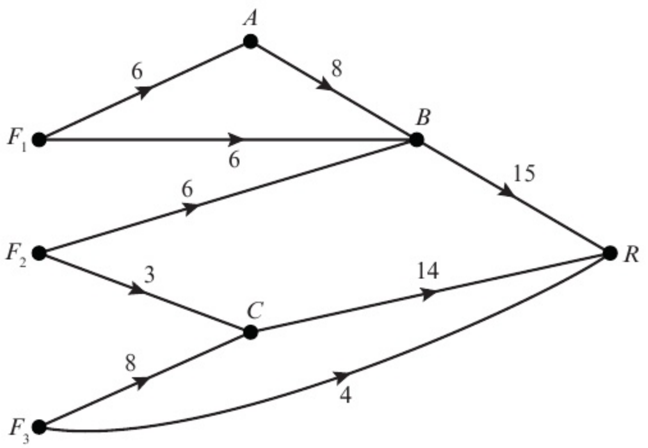
**e** Calculate the new maximum flow through the system. **(3 marks)**

**f** Show the new maximum flow pattern on a diagram. **(2 marks)**

## Mixed exercise 4

**Answer templates for questions marked * are available at www.pearsonschools.co.uk/d2maths**

E **1*** A company wishes to transport its products from three factories $F_1$, $F_2$ and $F_3$ to a single retail outlet $R$. The capacities of the possible routes, in van loads per day, are shown.




**a** Add a supersource $S$ to obtain a capacitated network with a single source and a single sink. State the minimum capacity of each arc you have added. **(2 marks)**

**b** **i** State the maximum flow along $SF_1ABR$ and $SF_3CR$. **(1 mark)**

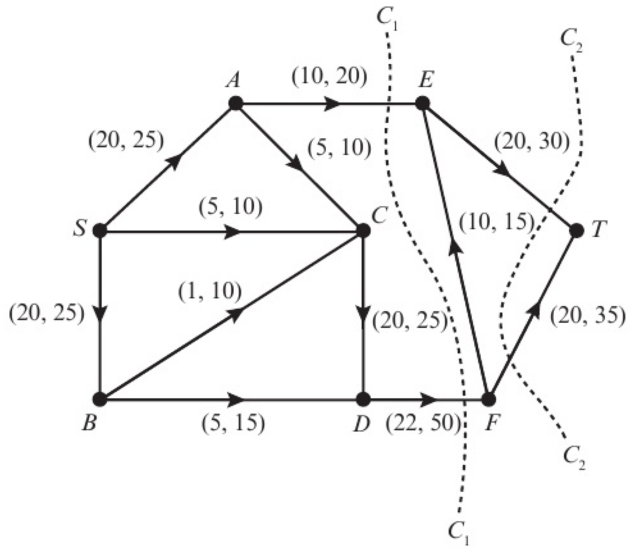
**ii** Show these maximum flows on a diagram, using numbers in circles. **(2 marks)**

Taking your answer to part **b ii** as the initial flow pattern,

**c** **i** use the labelling procedure to find a maximum flow from $S$ to $R$. List each flow-augmenting route you find, together with its flow. **(3 marks)**

**ii** Prove that your final flow is maximal. **(2 marks)**

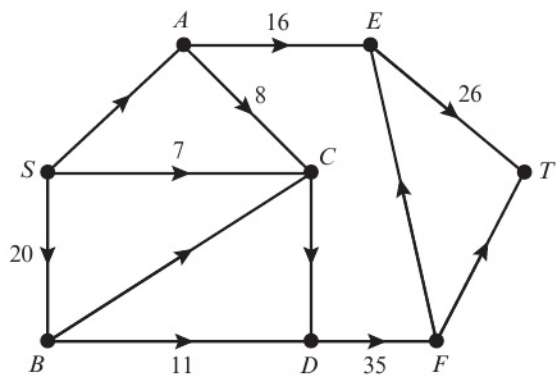
**2*** The diagram shows a capacitated directed network. The numbers on each arc represent lower and upper capacities.




**a** Give the value of each of the cuts, $C_1$ and $C_2$. **(2 marks)**

**b** Using your answers to part **a**, make a statement about the maximum flow through the network. **(2 marks)**

**c** Copy and complete the following diagram to show a feasible flow through the network. State the value of the flow. **(3 marks)**

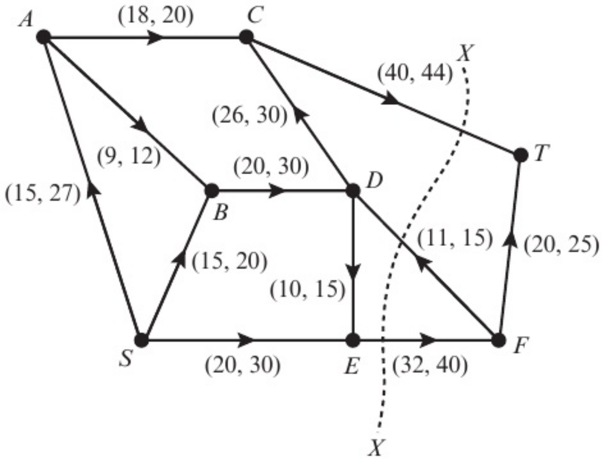



**d** Use flow augmentation to find the maximum flow. State each flow-augmenting path and its value. **(4 marks)**

**e** Find a cut with the same value as the maximum flow. **(1 mark)**

A **3*** The diagram shows a capacitated directed network. The numbers on each arc represent lower and upper capacities.

E

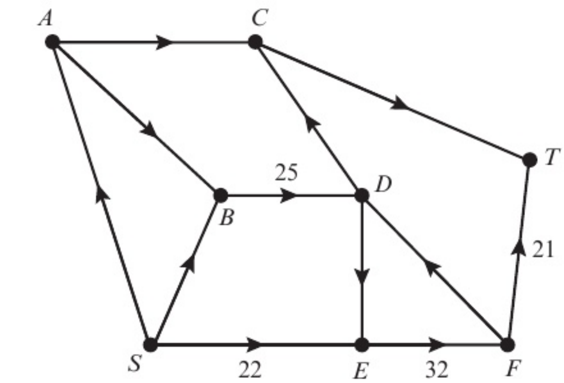



**a** Explain why the flows through $AC$ and $CD$ must be at their lowest values. **(2 marks)**

**b** Find the value of the flow through $AB$. **(1 mark)**

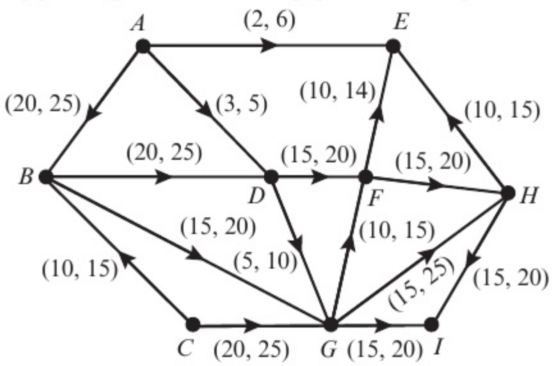
**c** Find the value of the cut, $X$. Explain what this means in terms of the maximum flow through the network. **(2 marks)**

**d** Copy and complete this diagram to show a feasible flow of 65 for the system. **(3 marks)**




**e** Find the value of the maximum flow and justify your answer. **(2 marks)**

E/P **4*** The diagram represents a system of pipes carrying oil. The numbers on each arc are the lower and upper capacities of the pipes in litres per second. The direction of flow is indicated by the arrows.




**a** Identify any sources and sinks. **(1 mark)**

A

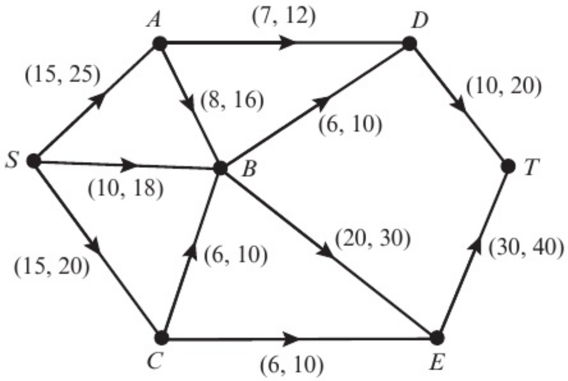
**b** Add a supersource and supersink to the diagram along with appropriately labelled arcs. **(2 marks)**

**c** Find the value of the cut through *AE*, *AD*, *BD*, *BG* and *BC*. Deduce what can be said about the maximum flow through the system. **(2 marks)**

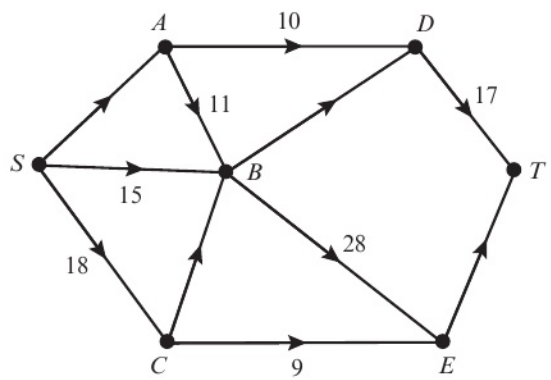
**d** The node at *F* is shut down for maintenance. By considering the node at *D*, explain why there is no feasible flow in the system. **(2 marks)**

**e** A new pipe is added with flow from *D* to *E*. State the minimum required capacity of this pipe. **(2 marks)**

E/P **5*** The diagram represents a capacitated directed network. The numbers on each arc are the lower and upper capacities of that arc.




**a** Copy and complete the following diagram to show a feasible flow through the network and state its value. **(3 marks)**




**b** Calculate the value of the flow through the node at *B*. **(1 mark)**

The capacity of the node at *B* is now reduced to 26.

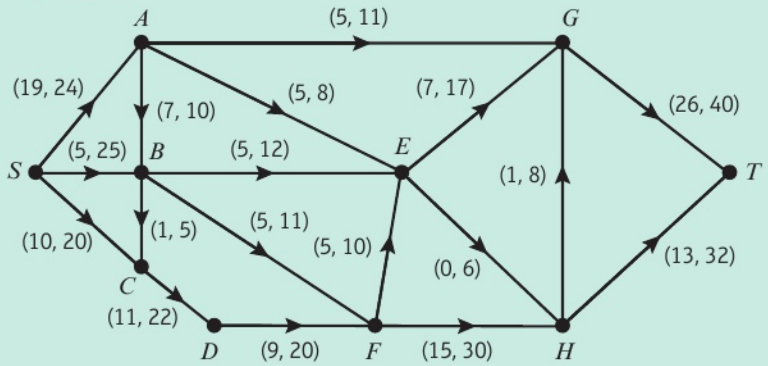
**c** Show this information on a diagram. **(1 mark)**

**d** Calculate the new maximum flow through the network and show the flow pattern on a diagram. **(3 marks)**

**e** Prove that the flow found in part **d** is maximal. **(3 marks)**

## Challenge

**A** The diagram shows a capacitated directed network. The numbers on the arcs represent lower and upper capacities.




**a** Find the maximum flow through this network. Show your flow pattern on a diagram.

**b** Prove that your flow is maximal.

One of the nodes becomes blocked and the maximum flow is then 39.

**c** Identify the blocked node and show a flow of 39 through the network.

**d** Prove that 39 is the maximum flow.

## Summary of key points

**1** In a capacitated directed network, if there are two capacities associated with each arc, the first number represents the **lower capacity** and the second number represents the **upper capacity**.

**2** If the maximum total flow into a vertex is equal to the minimum total flow out of that vertex, then:
- arcs into that vertex are at their upper capacities (saturated)
- arcs out of that vertex are at their lower capacities

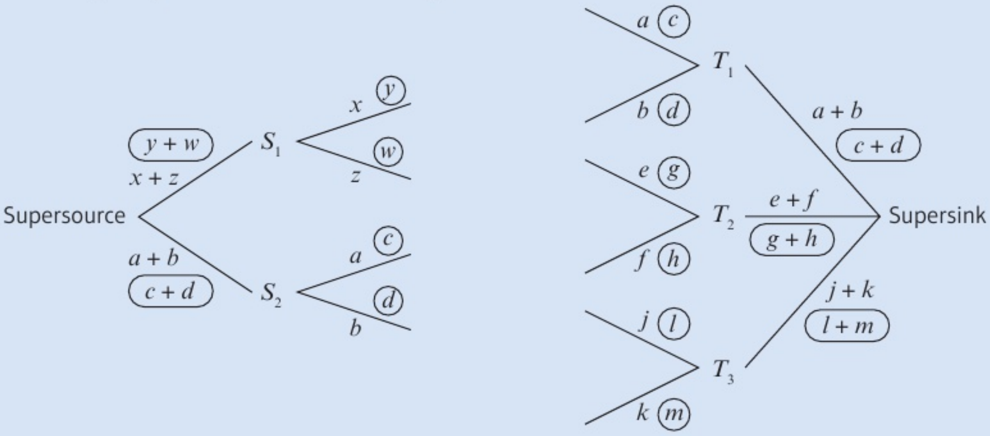
**3** If the minimum total flow into a vertex is equal to the maximum total flow out of that vertex, then:
- arcs into that vertex are at their lower capacities
- arcs out of that vertex are at their upper capacities (saturated).

**4** If a network includes lower capacities, the capacity (value) of a cut is equal to the sum of the upper capacities crossing the cut from $S$ to $T$, minus the sum of the lower capacities crossing the cut from $T$ to $S$.

A

**5** The arcs leading from the **supersource** to each source must have capacity and flow equal to the total capacity and total flow leaving that source.

The arcs leading from each sink to the **supersink** must have capacity and flow equal to the total capacity and total flow arriving at that sink.




**6** Each arc from the supersource to a source must have a lower capacity equal to the sum of the lower capacities of the arcs leaving the source and an upper capacity equal to the sum of the upper capacities of the arcs leaving the source.

Each arc from a sink to the supersink must have a lower capacity equal to the sum of the lower capacities of the arcs entering the sink and an upper capacity equal to the sum of the upper capacities of the arcs entering the sink.

**7** If a node (vertex) has restricted capacity then, on the network diagram, it may be replaced by two nodes joined by an arc of that capacity. Flow problems for the network can then be solved in the usual way.

**8** In the special case where a node is completely blocked, the node and any arcs linked directly to it may be removed from the network.

# Review exercise 1

**Answer templates for questions marked * are available at www.pearsonschools.co.uk/d2maths**

A E **1** The table shows the cost of transporting one unit of stock from each of three warehouses $W_1$, $W_2$, $W_3$ to each of three factories $F_1$, $F_2$ and $F_3$. It also shows the stock held at each warehouse and the amount required by each factory. The total number of units available is equal to the number of units required.

| | $W_1$ | $W_2$ | $W_3$ | Supply |
|---|---|---|---|---|
| $F_1$ | 7 | 8 | 6 | 4 |
| $F_2$ | 9 | 2 | 4 | 3 |
| $F_3$ | 5 | 6 | 3 | 8 |
| **Demand** | 2 | 9 | 4 | |

**a** Use the north-west corner method to obtain a possible pattern of distribution and find its cost. **(2)**

**b** Calculate the shadow costs and improvement indices. **(5)**

**c** Using your answer to part **b**, explain why the pattern is optimal. **(1)**

← Sections 1.1, 1.3

E **2** The following minimising transportation problem is to be solved.

The table shows the cost of transporting one unit of stock from each of three supply points $A$, $B$ and C to each of two demand points $J$ and $K$. It also shows the stock held at each supply point and the amount required at each demand point.

| | $J$ | $K$ | Supply |
|---|---|---|---|
| $A$ | 12 | 15 | 9 |
| $B$ | 8 | 17 | 13 |
| $C$ | 4 | 9 | 12 |
| **Demand** | 9 | 11 | |

A **a** Explain why a dummy demand point is needed. **(1)**

A possible north-west corner solution using a dummy demand point $L$ is:

| | $J$ | $K$ | $L$ |
|---|---|---|---|
| $A$ | 9 | 0 | |
| $B$ | | 11 | 2 |
| $C$ | | | 12 |

**b** Explain why it was necessary to place a zero in the first row of the second column. **(1)**

After three iterations of the stepping-stone method the table becomes:

| | $J$ | $K$ | $L$ |
|---|---|---|---|
| $A$ | | 8 | 1 |
| $B$ | | | 13 |
| $C$ | 9 | 3 | |

**c** Taking the most negative improvement index as the entering cell for the stepping-stone method, solve the transportation problem. You must make your shadow costs and improvement indices clear, and demonstrate that your solution is optimal. **(5)**

← Sections 1.1, 1.2, 1.3

E **3** Freezy Co. has three factories $A$, $B$ and $C$. It supplies freezers to three shops $D$, $E$ and $F$. The table shows the transportation cost in pounds of moving one freezer from each factory to each outlet. It also shows the number of freezers available for delivery at each factory and the number of freezers required at each shop. The total number of freezers required is equal to the total number of freezers available.

A

| | $D$ | $E$ | $F$ | Supply |
|---|---|---|---|---|
| $A$ | 21 | 24 | 16 | 24 |
| $B$ | 18 | 23 | 17 | 32 |
| $C$ | 15 | 19 | 25 | 14 |
| Demand | 20 | 30 | 20 | |

**a** Use the north-west corner method to find an initial solution. **(2)**

**b** Obtain improvement indices for each unused route. **(5)**

**c** Use the stepping-stone method **once** to obtain a better solution and state its cost. **(4)**

← Sections 1.1, 1.3, 1.4

E

**4** The manager of a car-hire firm has to arrange to move cars from three garages $A$, $B$ and $C$ to three airports $D$, $E$ and $F$ so that customers can collect them. The table below shows the transportation cost in pounds of moving one car from each garage to each airport. It also shows the number of cars available in each garage and the number of cars required at each airport. The total number of cars available is equal to the total number required.

| | Airport $D$ | Airport $E$ | Airport $F$ | Cars available |
|---|---|---|---|---|
| Garage $A$ | 20 | 40 | 10 | 6 |
| Garage $B$ | 20 | 30 | 40 | 5 |
| Garage $C$ | 10 | 20 | 30 | 8 |
| Cars required | 6 | 9 | 4 | |

**a** Use the north-west corner method to obtain a possible pattern of distribution and find its cost. **(2)**

**b** Calculate shadow costs for this pattern and hence obtain improvement indices for each route. **(5)**

**c** Use the stepping-stone method to obtain an optimal solution and state its cost. **(4)**

← Sections 1.1, 1.3, 1.4

A E

**5** A steel manufacturer has 3 factories $F_1$, $F_2$ and $F_3$ which can produce 35, 25 and 15 kilotonnes of steel per year, respectively. Three businesses $B_1$, $B_2$ and $B_3$ have annual requirements of 20, 25 and 30 kilotonnes respectively. The table below shows the cost $C_{ij}$, in appropriate units, of transporting one kilotonne of steel from factory $F_i$ to business $B_j$.

| | | Business | | |
|---|---|---|---|---|
| | | $B_1$ | $B_2$ | $B_3$ |
| Factory | $F_1$ | 10 | 4 | 11 |
| | $F_2$ | 12 | 5 | 8 |
| | $F_3$ | 9 | 6 | 7 |

The manufacturer wishes to transport the steel to the businesses at minimum total cost.

**a** Write down the transportation pattern obtained by using the north-west corner method. **(1)**

**b** Calculate all of the improvement indices $I_{ij}$. **(4)**

**c** Use the stepping-stone method to obtain an improved solution. **(3)**

**d** Show that the transportation pattern obtained in part **c** is optimal and find its cost. **(3)**

← Sections 1.1, 1.3, 1.4

E

**6 a** Describe a practical problem that could be solved using the transportation algorithm. **(1)**

A problem is to be solved using the transportation algorithm. The costs are shown in the table. The supply is from $A$, $B$ and $C$ and the demand is at $D$ and $E$.

| | $D$ | $E$ | Supply |
|---|---|---|---|
| $A$ | 5 | 3 | 45 |
| $B$ | 4 | 6 | 35 |
| $C$ | 2 | 4 | 40 |
| Demand | 50 | 60 | |

**b** Explain why it is necessary to add a third demand point $F$. **(2)**

A

**c** Use the north-west corner method to obtain a possible pattern of distribution and find its cost. **(4)**

**d** Calculate shadow costs and improvement indices for this pattern. **(4)**

**e** Use the stepping-stone method once to obtain an improved solution and its cost. **(4)**

← Sections 1.1, 1.2, 1.3, 1.4

E/P **7** A transportation company supplies televisions from manufacturers *A*, *B*, *C* and *D* to retail outlets *P*, *Q*, *R* and *S*.

The table shows the transportation cost per television, in pounds, from each manufacturer to each retail outlet. It also shows the availability of stock and the demand to be met. A minimum cost solution is required.

| | *P* | *Q* | *R* | *S* | Supply |
|---|---|---|---|---|---|
| ***A*** | 5 | 6 | 8 | 10 | 6 |
| ***B*** | 9 | 6 | 7 | 11 | 7 |
| ***C*** | 8 | 9 | 7 | 9 | 5 |
| ***D*** | 10 | 11 | 8 | 8 | 8 |
| **Demand** | 7 | 5 | 6 | 8 | |

**a** Use the north-west corner method to obtain an initial solution. **(2)**

**b** Solve the problem using the transportation algorithm. At each stage:

- show the shadow costs and improvement indices
- show the route
- state the entering cell and exiting cell.

Continue until an optimal solution is found and explain how you know that it is optimal. **(10)**

← Sections 1.1, 1.3, 1.4

E/P **8** A coach company has 20 coaches. At the end of a given week, 8 coaches are at depot *A*, 5 coaches are at depot *B* and 7 coaches are at depot *C*. At the beginning of the next week, 4 of these coaches are required at depot *D*, 10 of them at depot *E* and 6 of them at depot *F*. The following table shows the distances, in miles, between the relevant depots.

| | *D* | *E* | *F* |
|---|---|---|---|
| ***A*** | 40 | 70 | 25 |
| ***B*** | 20 | 40 | 10 |
| ***C*** | 35 | 85 | 15 |

The company needs to move the coaches between depots at the weekend. The total mileage covered is to be a minimum. This information is to be formulated as a linear programming problem.

**a** State clearly the decision variables. **(2)**

**b** Write down the objective function in terms of your decision variables. **(2)**

**c** Write down the constraints, explaining what each constraint represents. **(3)**

← Section 1.5

E/P **9** Three warehouses *W*, *X* and *Y* supply televisions to three supermarkets *J*, *K* and *L*. The table gives the cost, in pounds, of transporting a television from each warehouse to each supermarket. The warehouses have stocks of 34, 57 and 25 televisions respectively, and the supermarkets require 20, 56 and 40 televisions respectively. The total cost of transporting the televisions is to be minimised.

| | *J* | *K* | *L* |
|---|---|---|---|
| ***W*** | 3 | 6 | 3 |
| ***X*** | 5 | 8 | 4 |
| ***Y*** | 2 | 5 | 7 |

Formulate this transportation problem as a linear programming problem. Make clear your decision variables, objective function and constraints. **(7)**

← Section 1.5

A E/P **10** Three depots, $F$, $G$ and $H$ supply petrol to three service stations, $S$, $T$ and $U$. The table gives the cost, in pounds, of transporting 1000 litres of petrol from each depot to each service station.

| | $S$ | $T$ | $U$ |
|---|---|---|---|
| $F$ | 23 | 31 | 46 |
| $G$ | 35 | 38 | 51 |
| $H$ | 41 | 50 | 63 |

$F$, $G$ and $H$ have stocks of 540 000, 789 000 and 673 000 litres respectively. $S$, $T$ and $U$ require 257 000, 348 000 and 410 000 litres respectively. The total cost of transporting the petrol is to be minimised.

Formulate this problem as a linear programming problem. Make clear your decision variables, objective function and constraints. **(7)**

← Section 1.5

E **11** A theme park has four sites, $A$, $B$, $C$ and $D$ on which to put kiosks. Each kiosk will sell a different type of refreshment. The income from each kiosk depends upon what it sells and where it is located. The table below shows the expected daily income, in pounds, from each kiosk at each site.

| | Hot dogs and beef burgers ($H$) | Ice cream ($I$) | Popcorn, candyfloss and drinks ($P$) | Snacks and hot drinks ($S$) |
|---|---|---|---|---|
| Site $A$ | 267 | 272 | 276 | 261 |
| Site $B$ | 264 | 271 | 278 | 263 |
| Site $C$ | 267 | 273 | 275 | 263 |
| Site $D$ | 261 | 269 | 274 | 257 |

Reducing rows first, use the Hungarian algorithm to determine a site for each kiosk in order to maximise the total income.

State the site for each kiosk and the total expected income. You must make your method clear and show the table after each stage. **(10)**

← Section 2.1

E **12** An engineering company has 4 machines available and 4 jobs to be completed. Each machine is to be assigned to one job. The time, in hours, required by each machine to complete each job is shown in the table below.

| | **Job 1** | **Job 2** | **Job 3** | **Job 4** |
|---|---|---|---|---|
| **Machine 1** | 14 | 5 | 8 | 7 |
| **Machine 2** | 2 | 12 | 6 | 5 |
| **Machine 3** | 7 | 8 | 3 | 9 |
| **Machine 4** | 2 | 4 | 6 | 10 |

Use the Hungarian algorithm, reducing rows first, to obtain the allocation of machines to jobs which minimises the total time required. State this minimum time. **(10)**

← Section 2.1

E **13** A large room in a hotel is to be prepared for a wedding reception. The tasks that need to be carried out are:

**I** clean the room

**II** arrange the tables and chairs

**III** set the places

**IV** arrange the decorations

The tasks need to be completed consecutively and the room must be prepared in the least possible time. The tasks are to be assigned to four teams of workers $A$, $B$, $C$ and $D$. Each team must carry out only one task. The table below shows the times, in minutes, that each team takes to carry out each task.

| | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| **I** | 17 | 24 | 19 | 18 |
| **II** | 12 | 23 | 16 | 15 |
| **III** | 16 | 24 | 21 | 18 |
| **IV** | 12 | 24 | 18 | 14 |

**a** Use the Hungarian algorithm to determine which team should be assigned to each task. You must make your method clear and show:

**i** the state of the table after each stage in the algorithm **(6)**

**ii** the final allocation. **(2)**

**b** Obtain the minimum total time taken for the room to be prepared. **(2)**

← Section 2.1

E **14** Six workers $A$, $B$, $C$, $D$, $E$ and $F$ are to be assigned to six tasks $P$, $Q$, $R$, $S$, $T$ and $U$. Each worker must be assigned to just one task and each task must have just one worker allocated to it.

The table shows the cost, in pounds, of assigning each worker to a given task.

| | $P$ | $Q$ | $R$ | $S$ | $T$ | $U$ |
|---|---|---|---|---|---|---|
| $A$ | 16 | 18 | 14 | 21 | 11 | 23 |
| $B$ | 19 | 20 | 17 | 19 | 10 | 22 |
| $C$ | 18 | 21 | 16 | 20 | 12 | 24 |
| $D$ | 15 | 19 | 17 | 20 | 10 | 21 |
| $E$ | 18 | 20 | 15 | 22 | 11 | 25 |
| $F$ | 17 | 24 | 16 | 23 | 14 | 28 |

**a** Reducing rows first, use the Hungarian algorithm to obtain an allocation that minimises the total cost. **(9)**

**b** Find the minimum cost. **(2)**

← Section 2.1

E **15** Four workers Beth, Josh, Louise and Oliver are to be assigned to three tasks 1, 2 and 3.

Each worker can be assigned to at most one task and each task is to be completed by just one worker.

The time taken, in minutes, by each worker to complete each task is shown in the table.

| | 1 | 2 | 3 |
|---|---|---|---|
| **Beth** | 15 | 18 | 24 |
| **Josh** | 16 | 15 | 23 |
| **Louise** | 15 | 17 | 22 |
| **Oliver** | 14 | 19 | 23 |

**a** Explain why a dummy column is needed to solve this problem using the Hungarian algorithm. **(1)**

**b** Reducing by rows first, use the Hungarian algorithm to allocate the workers to tasks so that the total time is minimised. **(7)**

**c** Find the minimum time required to complete the three tasks. **(1)**

← Section 2.2

E **16** Three car mechanics Bill, Steve and Jo are to be sent for training on servicing hybrid vehicles. Courses are available at four different centres $A$, $B$, $C$ and $D$. Costs include travel from home and accommodation. At most one mechanic can be allocated to each centre.

The table shows the cost, in pounds, of assigning each mechanic to each centre.

| | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| **Bill** | 570 | 375 | 440 | 520 |
| **Steve** | 510 | 420 | 480 | 470 |
| **Jo** | 550 | 395 | 410 | 490 |

**a** Reducing rows first, use the Hungarian algorithm to allocate the mechanics to training centres so that the total cost is minimised. **(8)**

**b** Find the total cost. **(2)**

← Section 2.2

E **17** In a quiz there are four individual rounds, Art, Literature, Music and Science. A team consists of four people, Donna, Hannah, Kerwin and Thomas. Each of the four rounds must be answered by a different team member. The table shows the number of points that each team member is likely to get on each individual round.

| | Art | Literature | Music | Science |
|---|---|---|---|---|
| **Donna** | 31 | 24 | 32 | 35 |
| **Kerwin** | 19 | 14 | 20 | 21 |
| **Hannah** | 16 | 10 | 19 | 22 |
| **Thomas** | 18 | 15 | 21 | 23 |

Use the Hungarian algorithm, reducing rows first, to obtain an allocation which maximises the total points likely to be scored in the four rounds. You must make your method clear and show the table after each stage. **(10)**

← Section 2.3

E/P **18** Talkalot College holds an induction meeting for new students. The meeting consists of four talks: I (Welcome), II (Options and Facilities), III (Study Tips) and IV (Planning for Success). The four department heads, Clive, Julie, Nicky and Steve, deliver one of these talks each. The talks are delivered consecutively and there are no breaks between talks. The meeting starts at 10 a.m. and ends when all four talks have been delivered. The time, in minutes, each department head takes to deliver each talk is given in the table below.

| | Talk I | Talk II | Talk III | Talk IV |
|---|---|---|---|---|
| **Clive** | 12 | 34 | 28 | 16 |
| **Julie** | 13 | 32 | 36 | 12 |
| **Nicky** | 15 | 32 | 32 | 14 |
| **Steve** | 11 | 33 | 36 | 10 |

**a** Use the Hungarian algorithm to find the earliest time that the meeting could end. You must make your method clear and show:

**i** the state of the table after each stage in the algorithm **(8)**

**ii** the final allocation. **(1)**

**b** Modify the table so it could be used to find the latest time that the meeting could end. (You do not have to find this latest time.) **(3)**

← Sections 2.1, 2.3

E/P **19** Four salespersons Ann, Brenda, Connor and Dave are to be sent to visit four companies 1, 2, 3 and 4. Each salesperson will visit exactly one company, and all companies will be visited.

Previous sales figures show that each salesperson will make sales of different values, depending on the company that they visit. These values (in £10 000s) are shown in the table below.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| **Ann** | 26 | 30 | 30 | 30 |
| **Brenda** | 30 | 23 | 26 | 29 |
| **Connor** | 30 | 25 | 27 | 24 |
| **Dave** | 30 | 27 | 25 | 21 |

**a** Use the Hungarian algorithm to obtain an allocation that maximises the sales. You must make your method clear and show the table after each stage. **(9)**

**b** State the value of the maximum sales. **(1)**

**c** Show that there is a second allocation that maximises the sales. **(2)**

← Section 2.3

E/P **20** A team of four runners Amy, Bob, Charles and Davina has entered for a 20 km relay race. There are four stages 1, 2, 3 and 4 to complete over different types of terrain. Each runner must complete just one stage.

The table shows the times, in minutes, taken by each runner over each stage in practice.

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| ***A*** | 21 | 19 | 23 | 20 |
| ***B*** | 20 | 18 | 21 | 19 |
| ***C*** | 22 | 20 | - | 21 |
| ***D*** | 20 | 19 | 22 | 20 |

Charles can't run stage 3 as it is over rough ground and may aggravate his knee injury.

**a** Reducing rows first, use the Hungarian algorithm to allocate each runner to a stage in order to minimise the total time. Show the table after each stage. **(9)**

**b** Find the minimum time needed to complete the course. **(1)**

← Section 2.4

A E/P **21** A manager wishes to purchase seats for a new cinema. He wishes to buy three types of seat: standard, deluxe and majestic. Let the number of standard, deluxe and majestic seats to be bought be $x$, $y$ and $z$ respectively.

He decides that the total number of deluxe and majestic seats should be at most half of the number of standard seats.

The number of deluxe seats should be at least 10% and at most 20% of the total number of seats.

The number of majestic seats should be at least half of the number of deluxe seats.

The total number of seats should be at least 250.

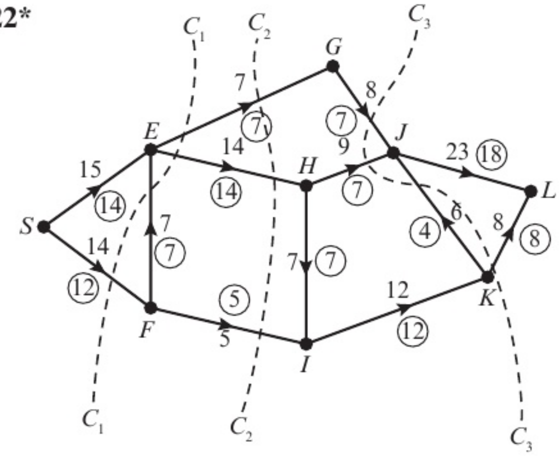
Standard, deluxe and majestic seats each cost £20, £26 and £36, respectively.

The manager wishes to minimise the total cost, £$C$, of the seats.

Formulate this situation as a linear programming problem, simplifying your inequalities so that all coefficients are integers. **(7)**

← Section 2.5

E/P **22***




The diagram shows a network of roads represented by arcs. The capacity of the road represented by that arc is shown on each arc. The numbers in circles represent a possible flow of 26 from $S$ to $L$.

Three cuts $C_1$, $C_2$ and $C_3$ are shown.

**a** Find the capacity of each of the three cuts. **(3)**

**b** Verify that the flow of 26 is maximal. **(2)**

The government aims to maximise the possible flow from $S$ to $L$ by using one of two options.

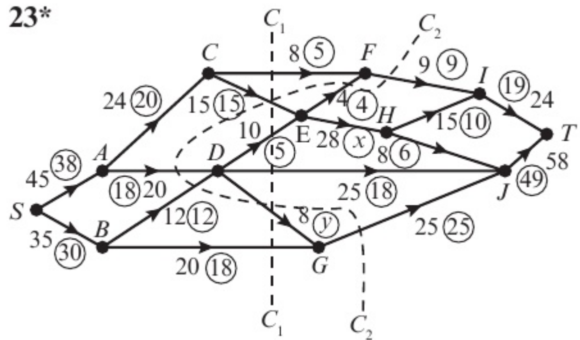
Option 1: Build a new road from $E$ to $J$ with capacity 5.

Option 2: Build a new road from $F$ to $H$ with capacity 3.

**c** By considering **both** options, explain which one meets the government's aim. **(5)**

← Section 3.2, 3.5

E **23***




The diagram shows a capacitated, directed network. The number on each arc indicates the capacity of that arc, and the numbers in circles show a feasible flow of value 68 through the network.

**a** Find the values of $x$ and $y$, explaining your method briefly. **(2)**

**b** Find the value of cuts $C_1$ and $C_2$. **(2)**

Starting with the given feasible flow of 68,

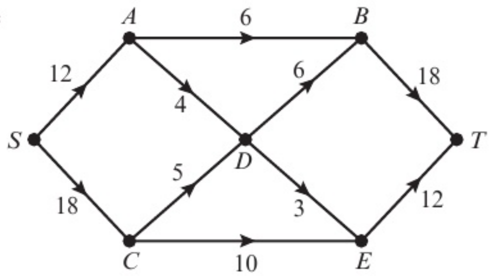
**c** use the labelling procedure to find a maximal flow through this network. List each flow-augmenting route you use, together with its flow. **(4)**

**d** Show your maximal flow and state its value. **(2)**

**e** Prove that your flow is maximal. **(2)**

← Section 3.2, 3.4, 3.5

(E/P) **24***



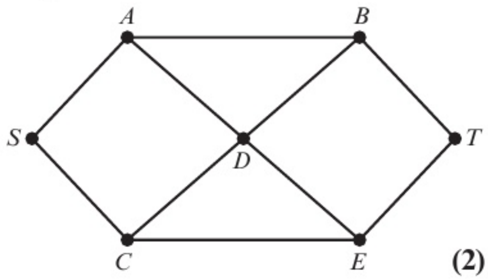

The diagram shows a capacitated directed network. The numbers on each arc indicate the capacity of that arc in appropriate units.

**a** Explain why it is not possible to achieve a flow of 30 through the network from $S$ to $T$. **(1)**

**b** State the maximum flow along:

**i** $SABT$ **ii** $SCET$ **(2)**

**c** Show these flows on a copy of the diagram below.




**(2)**

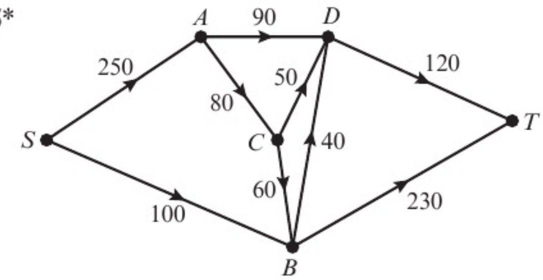
**d** Taking your answer to part **c** as the initial flow pattern, use the labelling procedure to find a maximum flow from $S$ to $T$. Show your working. List each flow-augmenting path you use together with its flow. **(3)**

**e** Indicate a maximum flow. **(1)**

**f** Prove that your flow is maximal. **(2)**

← Sections 3.4, 3.5

(E) **25***



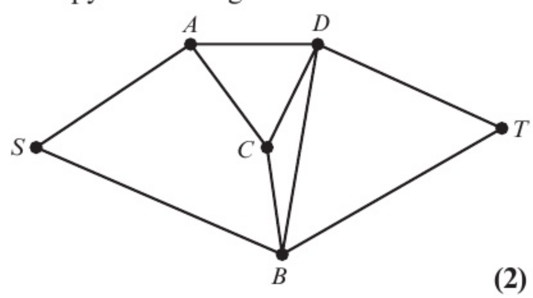

Natural gas is produced at $S$ and is transported to a refinery at $T$ by a network of underwater pipelines. The diagram shows the network of pipelines. The numbers on each arc represent the capacity of that arc in appropriate units.

**a** State the maximum flow along:

**i** $SACDT$

**ii** $SBT$ **(2)**

**b** Show these two maximum flows on a copy of the diagram below.




**(2)**

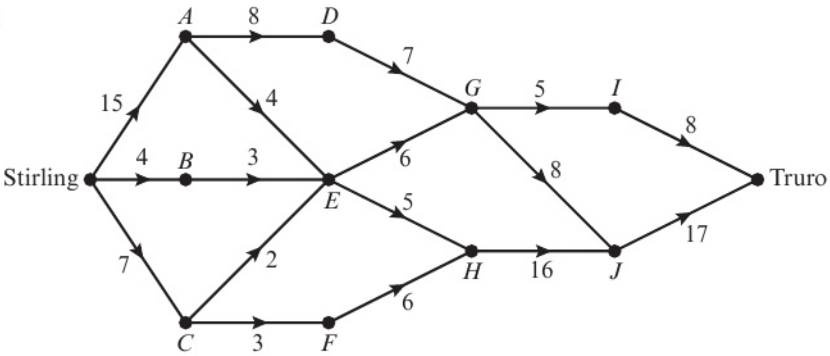
**c** Taking your answer to part **b** as the initial flow pattern, use the labelling procedure to find a maximum flow from $S$ to $T$ showing your working. List each flow-augmenting route you find and state its flow. **(3)**

**d** Draw your final flow pattern. **(1)**

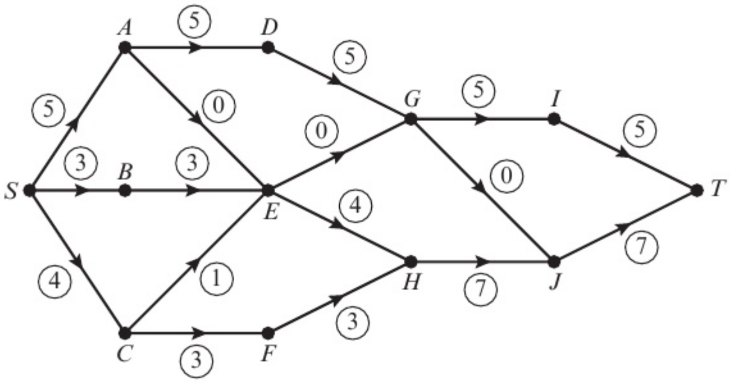
**e** Prove that your flow is maximal. **(2)**

← Sections 3.4, 3.5

E/P **26***




Twenty students wish to travel by bus from Stirling to Truro.
The network above shows various routes and the number of free seats available on the coaches that travel on these routes between the two cities. The students agree to travel singly or in small groups and meet up again in Truro.




The network above shows how 12 of the students could travel to Truro.

**a** Using this as your initial flow pattern and listing each flow-augmenting route you use, together with its flow, use the labelling procedure to find the maximum flow. **(4)**

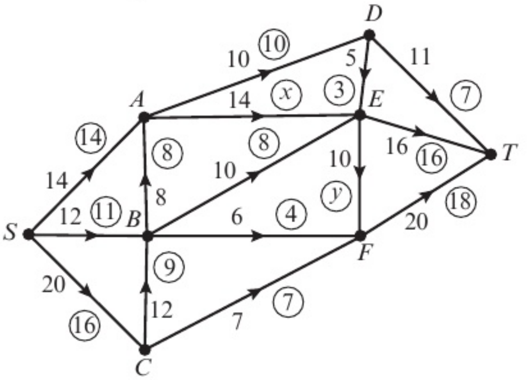
**b** Draw a network showing the maximum flow. **(2)**

**c** State how many students can travel from Stirling to Truro along these routes. **(2)**

**d** Verify your answer using the maximum flow – minimum cut theorem, listing the arcs that your minimum cut passes through. **(2)**

← Sections 3.4, 3.5

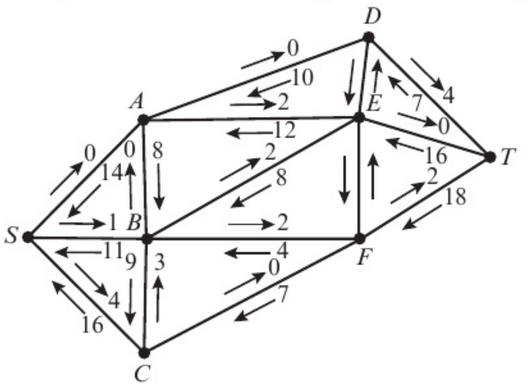
E/P **27*** The diagram shows a capacitated directed network. The number on each arc represents its capacity and the numbers in circles represent an initial flow.




**a** Calculate the value of $x$ and the value of $y$. **(2)**

**b** State the value of the initial flow. **(1)**

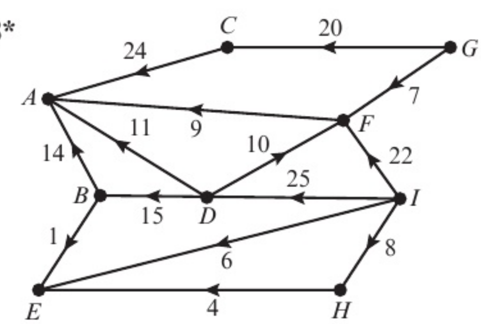
**c** Copy and complete the initial labelling procedure for the diagram below. **(3)**



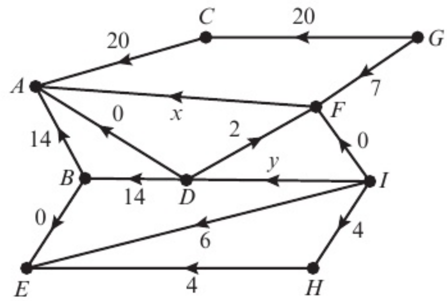

**d** Use flow augmentation to find the maximum flow. State each flow-augmenting route used. **(3)**

**e** Prove that the flow is maximal. **(3)**

A E **28***




The diagram above shows a capacitated directed network. The number on each arc is its capacity.




The diagram above shows a feasible initial flow through the same network.

**a** Calculate the value of $x$ and the value of $y$. **(2)**

**b** Obtain the value of the initial flow through the network, and explain how you know it is not maximal. **(2)**

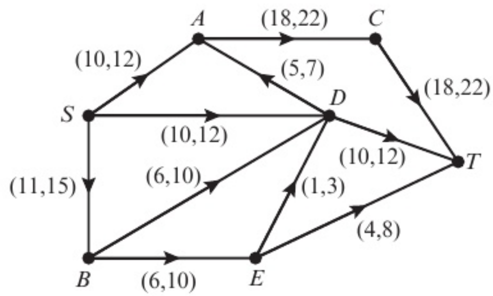
**c** Use this initial flow and the labelling procedure to find a maximum flow through the network. You must list each flow-augmenting route you use, together with its flow. **(3)**

**d** Show your maximal flow pattern. **(2)**

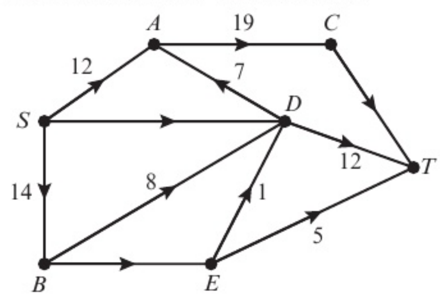
**e** Prove that your flow is maximal. **(2)**

← Section 4.1

E **29*** The diagram shows a capacitated directed network. The numbers on each arc represent the lower and upper capacity of the arc.




This diagram shows a partly completed feasible flow for the network.



A

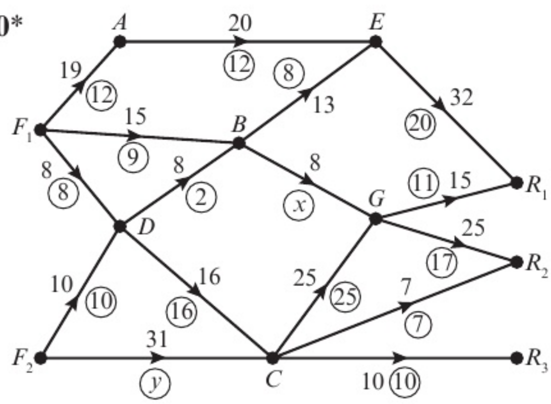
**a** Calculate each of the missing values and the flow through the network. **(3)**

**b** Use flow augmentation to maximise the flow through the network. Give each flow-augmenting path and state its value. State the value of the maximum flow. **(4)**

**c** Prove that the flow is maximal. **(2)**

← Section 4.1

E **30***




The diagram shows a capacitated directed network of pipes flowing from two oil fields $F_1$ and $F_2$ to three refineries $R_1$, $R_2$ and $R_3$. The number on each arc represents the capacity of the pipe and the numbers in circles represent a possible flow of 65.

**a** Find the value of $x$ and the value of $y$. **(2)**

**b** On a copy of the diagram, add a supersource and a supersink, and arcs showing their minimum capacities. **(2)**

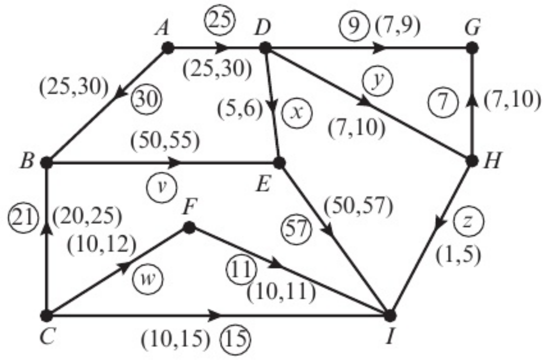
**c** Taking the given flow of 65 as the initial flow pattern, use the labelling procedure to find the maximum flow. State clearly your flow-augmenting routes. **(3)**

**d** Show the maximum flow and write down its value. **(2)**

**e** Prove that the flow is maximal. **(2)**

← Section 4.2

A E **31*** The diagram shows a capacitated directed network. The numbers in brackets on each arc represent the lower and upper capacity of the arc. The numbers in circles represent a feasible flow.




**a** On a copy of the diagram, add a supersource, $S$, and a supersink, $T$, to the diagram. Label the new arcs with their upper and lower capacities. **(2)**

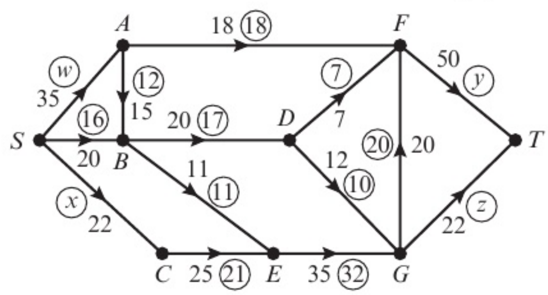
**b** Find the values of $v$, $w$, $x$, $y$ and $z$. **(4)**

**c** Calculate the flow through the network. **(2)**

**d** Prove that the flow is maximal. **(2)**

← Section 4.2

E/P **32*** The diagram represents a system of sewage pipes. The number on each pipe represents its capacity in litres/second and the numbers in circles represent the current values of the flow in each pipe.




**a** Find the value of the flow through the system and show that it is maximal. **(4)**

The Junction at $D$ is closed for maintenance.

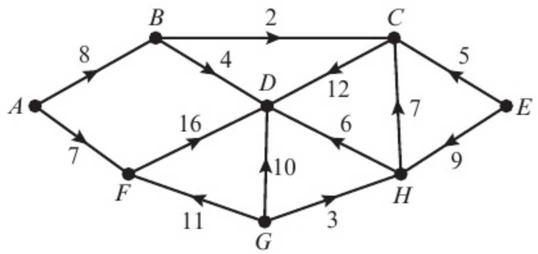
**b** Draw a diagram showing the new maximal flow pattern. **(3)**

A

**c** State the value of the flow and prove that it is maximal. **(3)**

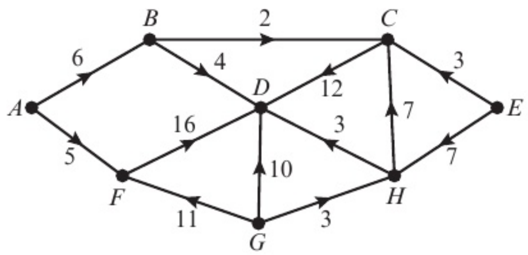
← Section 4.3

E **33*** The diagram below models a drainage system. The number on each arc indicates the capacity of that arc, in litres per second.




**a** Write down the source vertices. **(1)**

This diagram shows a feasible flow through the same network.




**b** State the value of the feasible flow shown. **(2)**

Taking the flow shown as your initial flow pattern,

**c** use the labelling procedure to find a maximum flow through this network. You should list each flow-augmenting route you use, together with its flow. **(3)**

**d** Show the maximum flow and state its value. **(2)**

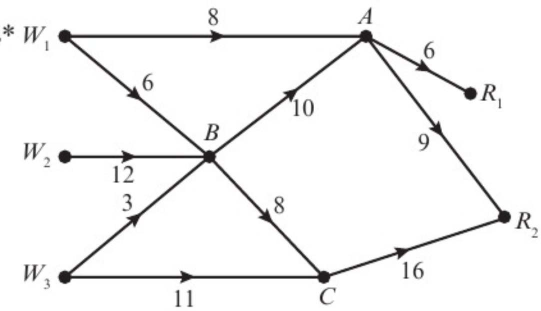
**e** Prove that your flow is maximal. **(2)**

**f** The flow through the junction at $C$ is restricted to 8 litres per second. Find the new maximum flow through the network. **(2)**

**g** Given that the flow through $EC$ is unchanged, show the new maximum flow on a diagram. **(2)**

← Sections 4.3

A E/P **34***




A company has three warehouses $W_1$, $W_2$ and $W_3$. It needs to transport the goods stored there to two retail outlets $R_1$ and $R_2$. The capacities of the possible routes, in van loads per day, are shown. Warehouses $W_1$, $W_2$ and $W_3$ have 14, 12 and 14 van loads respectively available per day and retail outlets $R_1$ and $R_2$ can accept 6 and 25 van loads respectively per day.

**a** On a copy of the diagram add a supersource $W$, a supersink $R$ and the appropriate directed arcs to obtain a single-source, single-sink capacitated network. State the minimum capacity of each arc you have added. **(2)**

**b** State the maximum flow along

**i** $W\ W_1\ A\ R_1\ R$,

**ii** $W\ W_3\ C\ R_2\ R$. **(2)**

**c** Taking your answers to part **b** as the initial flow pattern, use the labelling procedure to obtain a maximum flow through the network from $W$ to $R$. Show your working. List each flow-augmenting route you use, together with its flow. **(4)**

**d** From your final flow pattern, determine the number of van loads passing through $B$ each day. **(2)**

The company has the opportunity to increase the number of van loads from one of the warehouses $W_1$, $W_2$, $W_3$ to $A$, $B$ or $C$.

**e** Determine how the company should use this opportunity so that it achieves a maximum flow. **(2)**

← Sections 4.3

## Challenge

A

**1** A company supplies chicken to a chain of four fried chicken restaurants from three different farms. The table shows the cost, in pounds, of moving one box of chicken from each farm to each restaurant. It also shows the supply of chicken available at each farm and the demand at each restaurant each week.

| | $P$ | $Q$ | $R$ | $S$ | **Supply** |
|---|---|---|---|---|---|
| $A$ | 2 | 6 | 3 | 5 | 140 |
| $B$ | 4 | 4 | 7 | 3 | 300 |
| $C$ | 4 | 8 | 5 | 2 | 200 |
| **Demand** | 120 | 160 | 150 | 220 | |

**a** Explain why the weekly demand cannot be fully met.

The company incurs penalty payments for each restaurant that does not meet its demand. The penalty payment for each restaurant, in pounds per box of unmet demand, is given in the table below.

| **Restaurant** | $P$ | $Q$ | $R$ | $S$ |
|---|---|---|---|---|
| **Penalty per box** | 3 | 1 | 3 | 4 |

The company wishes to minimise its total weekly costs, including transportation costs and penalty payments.

**b** Find an optimal solution, stating the total cost of this solution.

← Sections 1.3, 1.4

**2** Four workers $A$, $B$, $C$, and $D$ are to be assigned to five tasks 1, 2, 3, 4 and 5.

One worker is to be assigned to two tasks but the other workers are to be assigned to just one task. Each task is to be completed by just one worker.

The table shows the cost, in pounds, of allocating each worker to each task.

Worker $C$ cannot do task 3 and worker $D$ cannot do task 1.

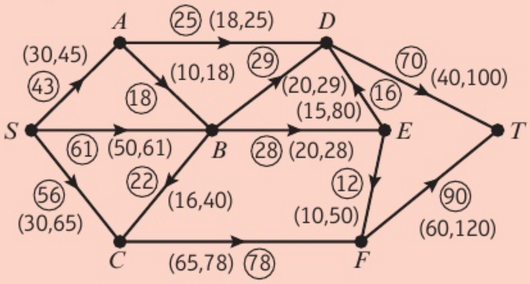
| | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| $A$ | 53 | 62 | 41 | 55 | 68 |
| $B$ | 59 | 55 | 57 | 46 | 60 |
| $C$ | 62 | 58 | - | 40 | 62 |
| $D$ | - | 59 | 64 | 49 | 58 |

Adapt the Hungarian algorithm to find an allocation which minimises the total cost.

← Section 2.4

A

**3*** The diagram shows a capacitated directed network that models a system of water pipes. The numbers in brackets represent lower and upper capacities and the numbers in circles represent a feasible flow.




**a** Show that the given flow is maximal.

The network is to be upgraded.

**b** Calculate the extra flow that could be achieved by replacing just one pipe with one of unlimited capacity. State which pipe should be upgraded for maximum effect.

**c** Calculate the extra flow that could be achieved by replacing two pipes with two of unlimited capacity. State which pipes should be upgraded for maximum effect.

← Section 4.1

# Dynamic programming 5

## Objectives

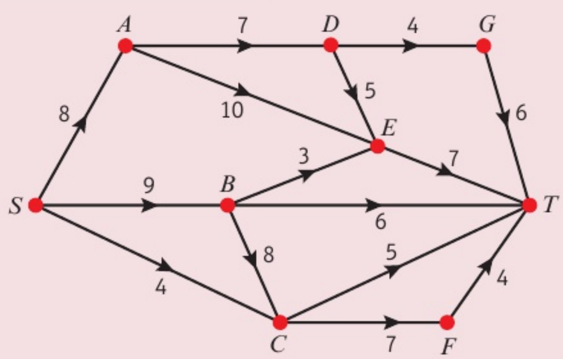
After completing this chapter you should be able to:

- Understand the terminology and principles of dynamic programming, including Bellman's principle of optimality → pages 152–161
- Use dynamic programming to solve maximum, minimum, minimax or maximin problems, presented in network form or table form → pages 161–174

As well as maximising or minimising the **total weight** of a path in a network, you might want to minimise the greatest weight of any one arc, or maximise the least weight of any one arc. You could use **minimax** (or **maximin**) conditions such as these to plan a kayaking route that avoids the most dangerous rapids.

→ Mixed exercise Q7

## Prior knowledge check




**a** Name an algorithm that could be used to find the path of least weight from $S$ to $T$.

**b** By inspection, or otherwise, find the route from $S$ to $T$ of least weight.

**c** Find the route of greatest weight from $S$ to $T$ that passes through $E$.

**d** Hence, write down the route of greatest weight from $S$ to $E$. ← D1, Section 3.4

## 5.1 Shortest and longest path problems

A

**Dynamic programming** is a technique for solving multi-stage decision-making problems. It involves a sequence of decisions, the object of which is to optimise time, profit, cost, or resources by taking the correct decision at each stage. It was first developed by Richard Bellman and others in the 1950s as a management tool and has been applied to production planning, machine scheduling, stock control, allocation of resources, maintenance and replacement of equipment, investment planning and process design, amongst others.

Some problems can be solved by drawing a directed network showing the possible decisions at each stage. You start at $S$ and can make one of several decisions. Depending upon which one you take, you can then take more decisions and so on, until you eventually reach $T$.

The easiest problems are about minimising the cost or time. This is, of course, the same as finding the shortest path from $S$ to $T$.

If the shortest/longest path from $S$ to $T$ is $SABCT$, then:
the shortest/longest path from $S$ to $C$ is $SABC$,
the shortest/longest path from $S$ to $B$ is $SAB$,
the shortest/longest path from $S$ to $A$ is $SA$,
and the shortest/longest path from $A$ to $C$ is $ABC$.

- **Any part of the shortest/longest path from $S$ to $T$ is itself a shortest/longest path.**

**Notation** This is known as **Bellman's principle of optimality** and is sometimes stated as 'any part of an optimal path is optimal'.

You will use the following terminology when finding optimal solutions to **network problems** using dynamic programming:

**Stage** The route from the initial state ($S$ say) to the final state ($T$ say) is made up of a sequence of moves. Each move is a stage. The stage tells you how 'far' you are from the destination vertex.

**State** This is the vertex that you are considering at each point.

**Action** This is the directed arc from one state to the next. In selecting an arc you are considering what happens if you do that action.

**Notation** In dynamic programming problems, **state** and **vertex** can be used interchangeably. The **initial state** is sometimes called the **source**, and the **final state** is sometimes called the **sink**.

**Destination** This is the vertex you arrive at having taken the action.

**Value** This is the sum of the weights on the arcs used in a sequence of actions.

When you use the shortest path algorithm from $S$ to $T$, you work forwards from S through the network, finding the shortest route from $S$ to every vertex.

- **In dynamic programming you work backwards from $T$ in a series of stages.**
  - **The vertices immediately before $T$ are examined. These are the stage 1 vertices. The best route from these to $T$ is noted.**

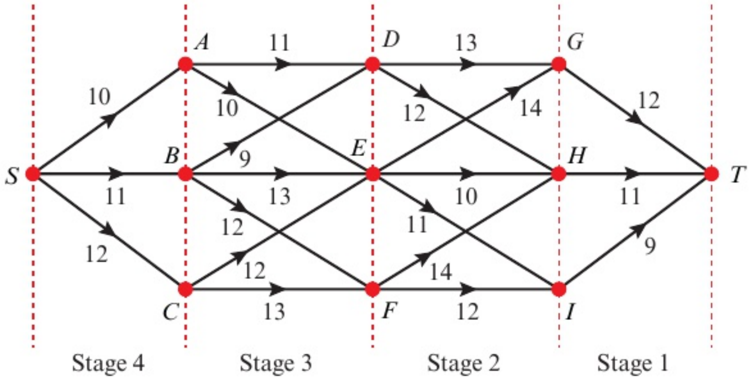
A

- **Then move back a stage, away from $T$ and towards $S$, to the stage 2 vertices. Routes from these vertices must pass through one of the stage 1 vertices to get to $T$. Use this to help you find the optimal route from each stage 2 vertex to $T$.**
- **Then move back a stage and repeat the process, until you reach $S$.**
  **The principle of optimality is used at each stage. The current optimal paths are developed using the paths found in the previous stage.**

You should be able to use dynamic programming to solve maximum and minimum problems, presented in network form.

## Example 1

**a** Use dynamic programming to find the minimum cost route from $S$ to $T$ in the network below.




**b** Hence write down the minimum cost route from $S$ to $I$.

**a** The solution is presented in the form of a table.

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| 1 | $G$ | $GT$ | $T$ | 12* |
| | $H$ | $HT$ | $T$ | 11* |
| | $I$ | $IT$ | $T$ | 9* |

Begin at $T$ so this is the first **stage**.
Stage 1
To get to $T$ you must pass through $G$, $H$ or $I$, so these are the **states**. In each case there is no choice of route and so the only **actions** are $GT$, $HT$ or $IT$. The **values** of these actions are the weights of the arcs 12, 11 and 9 respectively.
So the states $G$, $H$, and $I$ are given values of 12, 11 and 9 respectively.
This completes the first stage.

Use * to indicate the optimal action from each state. Note that in this case there is only one action for each state, so it must be optimal.

Now move back a stage.

A

| | | | | |
|---|---|---|---|---|
| 2 | $D$ | $DG$ | $G$ | 13 + 12 = 25 |
| | | $DH$ | $H$ | 12 + 11 = 23* |
| | $E$ | $EG$ | $G$ | 14 + 12 = 26 |
| | | $EH$ | $H$ | 10 + 11 = 21 |
| | | $EI$ | $I$ | 11 + 9 = 20* |
| | $F$ | $FH$ | $H$ | 14 + 11 = 25 |
| | | $FI$ | $I$ | 12 + 9 = 21* |
| 3 | $A$ | $AD$ | $D$ | 11 + 23 = 34 |
| | | $AE$ | $E$ | 10 + 20 = 30* |
| | $B$ | $BD$ | $D$ | 9 + 23 = 32* |
| | | $BE$ | $E$ | 13 + 20 = 33 |
| | | $BF$ | $F$ | 12 + 21 = 33 |

Stage 2
The stage 2 **states** are $D$, $E$ and $F$. You need to find the shortest route from each vertex to $T$.
From $D$ you must go to $DG$ or $DH$. These are the only two **actions** from state $D$.
From $E$ you must go to $EG$ or $EH$ or $EI$.
Finally, from $F$ you must go to $FH$ or $FI$.

You know the shortest route from each stage 1 vertex. You found them in the previous stage.
To find the total length of your route so far, add the weights.
From $D$ the actions and values are:
action $DG$ value = 13 ($DG$) + 12 ($GT$) = 25
action $DH$ value = 12 ($DH$) + 11 ($HT$) = 23
The smaller of these is 23 and this is indicated by *. This means that you can get from $D$ to $T$ in a minimum of 23.

From $E$ the actions are $EG$, $EH$ and $EI$.
Calculate the values, and indicate the smallest value with *.

From $F$ the actions are $FH$ and $FI$.
Calculate the values, and indicate the smaller value with *.

**Watch out** Now move back a stage. When using dynamic programming, you always **move backwards** through a problem.

Stage 3
The stage 3 **states** are $A$, $B$ and $C$.
From $A$ you must go to $AD$ or $AE$. These are the only two **actions** from state $A$.
From $B$ you must go to $BD$ or $BE$ or $BF$. These are the three **actions** from state $B$.
Finally from $C$, you must go to $CE$ or $CF$, giving the two **actions** from state $C$.

Calculate the values for states $A$, $B$ and $C$, and indicate the smallest values for each state with *.
For example, for state $A$:
action $AD$ value = 11($AD$) + 23($DT$) = 34
action $AE$ value = 10($AE$) + 20($ET$) = 30
So you can get from $A$ to $T$ in a minimum of 30.

| | | | | |
|---|---|---|---|---|
| | $C$ | $CE$ | $E$ | 12 + 20 = 32* |
| | | $CF$ | $F$ | 13 + 21 = 34 |
| 4 | $S$ | $SA$ | $A$ | 10 + 30 = 40* |
| | | $SB$ | $B$ | 11 + 32 = 43 |
| | | $SC$ | $C$ | 12 + 32 = 44 |

The shortest route is $SAEIT$ of value 40.

**b** The shortest route from $S$ to $T$ is $SAEIT$, which includes $I$. By Bellman's principle of optimality, the shortest route from $S$ to $I$ is $SAEI$.

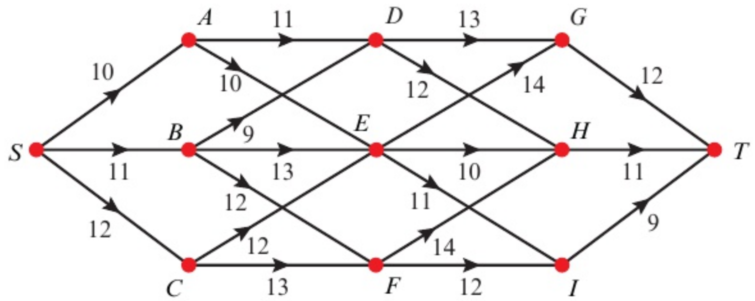
Now move back a stage.

Stage 4
From $S$ the actions and values are:
action $SA$ value = 10 ($SA$) + 30 ($AT$) = 40
action $SB$ value = 11 ($SB$) + 32 ($BT$) = 43
action $SC$ value = 12 ($SC$) + 32 ($CT$) = 44
The smallest of these is 40 and this is indicated by *.

To get the value of the route you look at the final stage and the optimal value is indicated by *.
To get the route, start at state $S$ and look for the *. This is on $SA$, so the route begins $SA$.
Now look at state $A$ for the *, this is on $AE$. So the route is $SAE$.
Now go to state $E$ and look for the *. It is on $EI$, so the route is $SAEI$.
Finally, look at state $I$ for the *. It is on $IT$, so the route is $SAEIT$.

**Example 2**

Use dynamic programming to find the maximum cost route from $S$ to $T$ in the network below.



| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| 1 | $G$ | $GT$ | $T$ | 12* |
| | $H$ | $HT$ | $T$ | 11* |
| | $I$ | $IT$ | $T$ | 9* |
| 2 | $D$ | $DG$ | $G$ | 13 + 12 = 25* |
| | | $DH$ | $H$ | 12 + 11 = 23 |
| | $E$ | $EG$ | $G$ | 14 + 12 = 26* |
| | | $EH$ | $H$ | 10 + 11 = 21 |
| | | $EI$ | $I$ | 11 + 9 = 20 |
| | $F$ | $FH$ | $H$ | 14 + 11 = 25* |
| | | $FI$ | $I$ | 12 + 9 = 21 |
| 3 | $A$ | $AD$ | $D$ | 11 + 25 = 36* |
| | | $AE$ | $E$ | 10 + 26 = 36* |
| | $B$ | $BD$ | $D$ | 9 + 25 = 34 |
| | | $BE$ | $E$ | 13 + 26 = 39* |
| | | $BF$ | $F$ | 12 + 25 = 37 |
| | $C$ | $CE$ | $E$ | 12 + 26 = 38* |
| | | $CF$ | $F$ | 13 + 25 = 38* |
| 4 | $S$ | $SA$ | $A$ | 10 + 36 = 46 |
| | | $SB$ | $B$ | 11 + 39 = 50* |
| | | $SC$ | $C$ | 12 + 38 = 50* |

There are three maximum routes, all of length 50:
$SBEGT$, $SCEGT$ and $SCFHT$

**Watch out** You need to find the **maximum** cost route. You will mark the **largest** value for each state with a *.

Notice that the values that are * in each stage are the ones you transfer to the next stage to calculate the value.

**Problem-solving**

At stage 3, actions $AD$ and $AE$ both produce paths of value 36. This means that both of these actions are optimal, so write a * next to each of them. Subsequent paths through $A$ can choose either of these actions.

Note that each vertex is used as a state and is used exactly once.

To find the route, just 'follow the stars'.

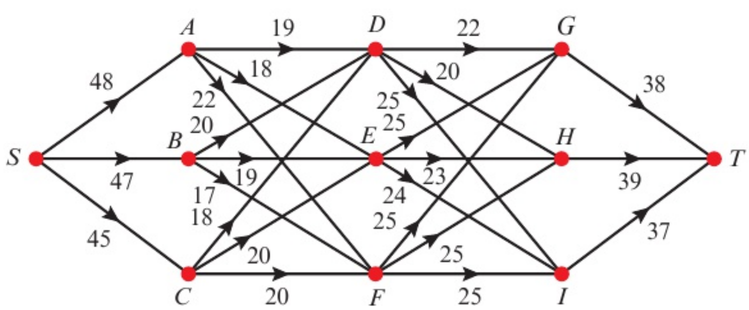
## Example 3

Use dynamic programming to find:

**a** the shortest

**b** the longest

route between $S$ and $T$. State the route and its length in each case.




**a** The shortest route

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| 1 | $G$ | $GT$ | $T$ | 38* |
| | $H$ | $HT$ | $T$ | 39* |
| | $I$ | $IT$ | $T$ | 37* |

A

| | | | | |
|---|---|---|---|---|
| 2 | $D$ | $DG$ | $G$ | 22 + 38 = 60 |
| | | $DH$ | $H$ | 20 + 39 = 59* |
| | | $DI$ | $I$ | 25 + 37 = 62 |
| | $E$ | $EG$ | $G$ | 25 + 38 = 63 |
| | | $EH$ | $H$ | 23 + 39 = 62 |
| | | $EI$ | $I$ | 24 + 37 = 61* |
| | $F$ | $FG$ | $G$ | 25 + 38 = 63 |
| | | $FH$ | $H$ | 25 + 39 = 64 |
| | | $FI$ | $I$ | 25 + 37 = 62* |
| 3 | $A$ | $AD$ | $D$ | 19 + 59 = 78* |
| | | $AE$ | $E$ | 18 + 61 = 79 |
| | | $AF$ | $F$ | 22 + 62 = 84 |
| | $B$ | $BD$ | $D$ | 20 + 59 = 79 |
| | | $BE$ | $E$ | 19 + 61 = 80 |
| | | $BF$ | $F$ | 17 + 62 = 79* |
| | $C$ | $CD$ | $D$ | 18 + 59 = 77* |
| | | $CE$ | $E$ | 20 + 61 = 81 |
| | | $CF$ | $F$ | 20 + 62 = 82 |
| 4 | $S$ | $SA$ | $A$ | 48 + 78 = 126 |
| | | $SB$ | $B$ | 47 + 79 = 126 |
| | | $SC$ | $C$ | 45 + 77 = 122* |

The shortest route length is 122. The route is $SCDHT$.

The route length is indicated by the * in the final stage.
To find the route, 'follow the stars'. Start with $SC$, indicated by *, which sends you to $C$. The optimal route from $C$ is $CD$ (as indicated by *). This sends you to $D$. From state $D$ you see that $DH$ has been indicated by *, which sends you to $H$. Finally $HT$ has been indicated by *. This gives you the route.

**b** The longest route

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| 1 | $G$ | $GT$ | $T$ | 38* |
| | $H$ | $HT$ | $T$ | 39* |
| | $I$ | $IT$ | $T$ | 37* |
| 2 | $D$ | $DG$ | $G$ | 22 + 38 = 60 |
| | | $DH$ | $H$ | 20 + 39 = 59 |
| | | $DI$ | $I$ | 20 + 39 = 62* |
| | $E$ | $EG$ | $G$ | 25 + 38 = 63* |
| | | $EH$ | $H$ | 23 + 39 = 62 |
| | | $EI$ | $I$ | 24 + 37 = 61 |
| | $F$ | $FG$ | $G$ | 25 + 38 = 63 |
| | | $FH$ | $H$ | 25 + 39 = 64* |
| | | $FI$ | $I$ | 25 + 37 = 62 |

| | | | | |
|---|---|---|---|---|
| 3 | $A$ | $AD$ | $D$ | 19 + 62 = 81 |
| | | $AE$ | $E$ | 18 + 63 = 81 |
| | | $AF$ | $F$ | 22 + 64 = 86* |
| | $B$ | $BD$ | $D$ | 20 + 62 = 82* |
| | | $BE$ | $E$ | 19 + 63 = 82* |
| | | $BF$ | $F$ | 17 + 64 = 81 |
| | $C$ | $CD$ | $D$ | 18 + 62 = 80 |
| | | $CE$ | $E$ | 20 + 63 = 83 |
| | | $CF$ | $F$ | 20 + 64 = 84* |
| 4 | $S$ | $SA$ | $A$ | 48 + 86 = 134* |
| | | $SB$ | $B$ | 47 + 82 = 129 |
| | | $SC$ | $C$ | 45 + 84 = 129 |

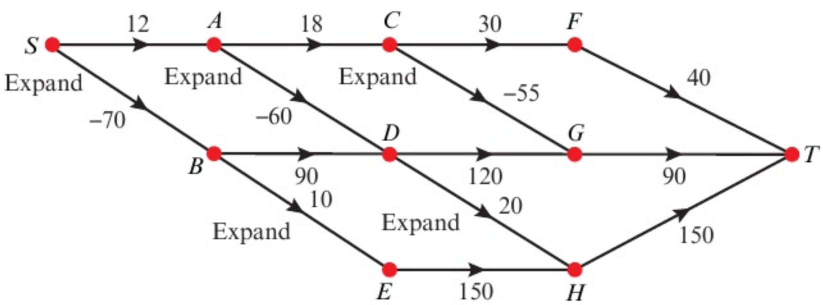
The longest route length is 134 and the route is $SAFHT$.

## Example 4

A firm is creating a three-year plan, and considering a possible expansion. If the firm expands there will be costs, due to purchasing new premises and equipment, but also increased revenue. To establish the effect of the decisions, the firm will then calculate the effect on selling all of its assets at the end of the three-year period.

The firm can only expand a maximum of two times due to plant restrictions.

The results of the decisions are shown in the network below, with negative numbers indicating costs and positive numbers indicating revenues, in £100 000s.




Determine the strategy that maximises the profit over the three-year period.

This is a maximisation problem. You are seeking a maximum route from $S$ to $T$.

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| 1 | $F$ | $FT$ | $T$ | 40* |
| (Sale of assets) | $G$ | $GT$ | $T$ | 90* |
| | $H$ | $HT$ | $T$ | 150* |

| | | | | |
|---|---|---|---|---|
| 2 (Third year) | C | CF | F | 30 + 40 = 70* |
| | | CG | G | −55 + 90 = 35 |
| | D | DG | G | 120 + 90 = 210* |
| | | DH | H | 20 + 150 = 170 |
| | E | EH | H | 150 + 150 = 300* |
| 3 (Second year) | A | AC | C | 18 + 70 = 88 |
| | | AD | D | −60 + 210 = 150* |
| | B | BD | D | 90 + 210 = 300 |
| | | BE | E | 10 + 300 = 310* |
| 4 (First year) | S | SA | A | 12 + 150 = 162 |
| | | SB | B | −70 + 310 = 240* |

The maximum profit is £24 000 000.

The route is *SBEHT*, which means:

| Year | Year 1 | Year 2 | Year 3 |
|---|---|---|---|
| Expand? | Expand | Expand | No expansion |

So the firm should expand in both year 1 and year 2 (and cannot expand in year 3).

In your final answer, express the value in pounds rather than hundreds of thousands of pounds. It is a clearer statement of the profit for non-mathematicians.

**Watch out** You are expected to be able to express your solutions in such a way that a non-mathematician could follow the instructions.

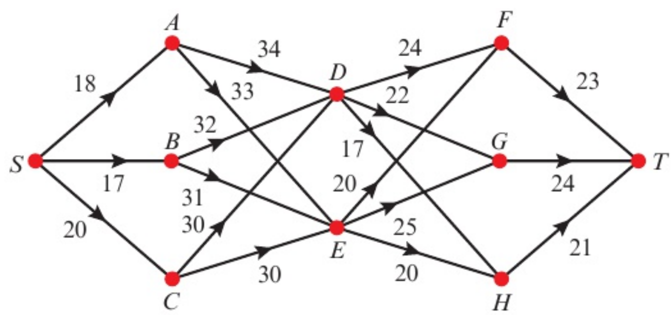
## Exercise 5A

**In questions 1 to 3**, use dynamic programming to find:

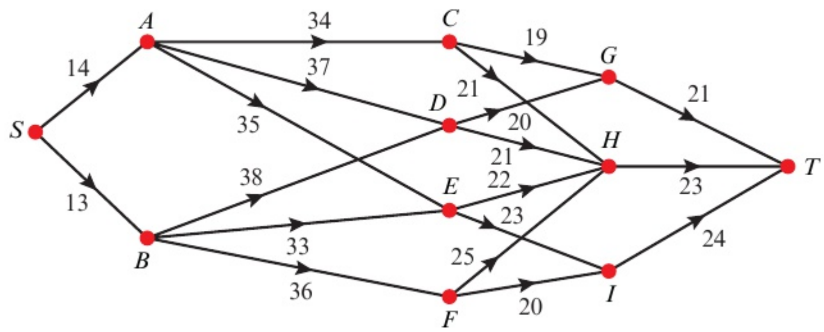
**a** a shortest **b** a longest

route from $S$ to $T$ in each network. State the route and its length in each case.

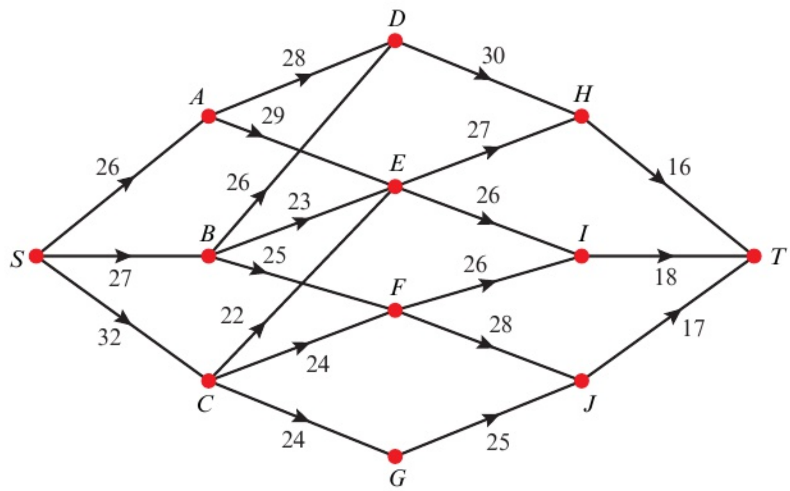
**1**



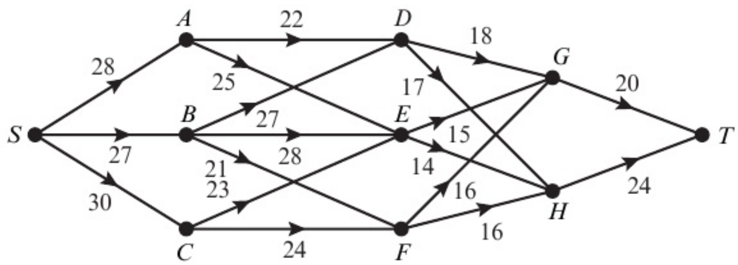

**2**



A **3**



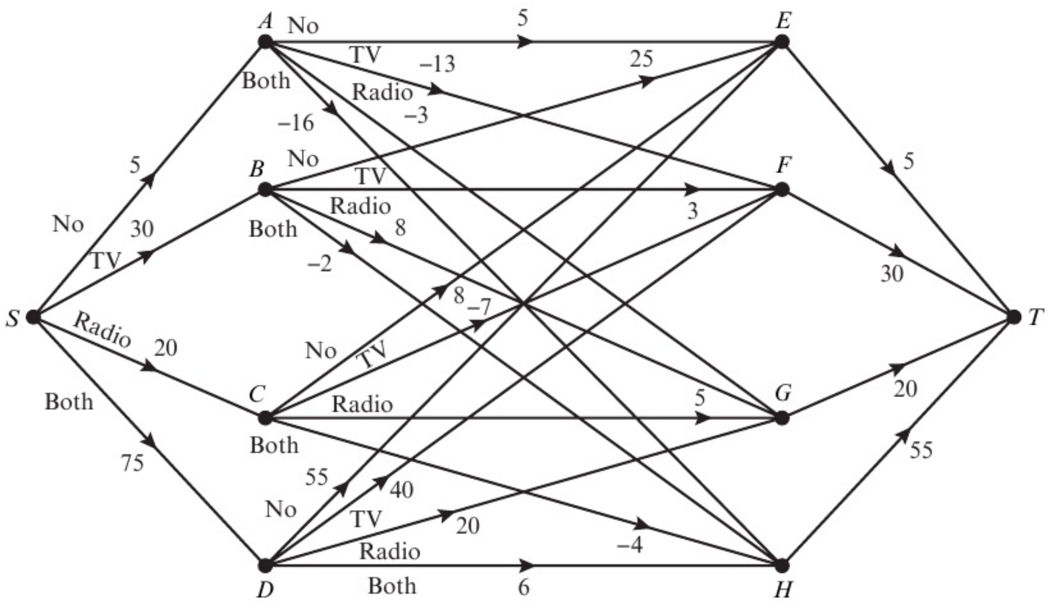

E **4** The network on the right represents the decisions associated with road maintenance in a small town over the next four years. The number on each arc represents the cost in £1000s corresponding to a particular decision.




**a** Use dynamic programming to determine which decisions the town should make for each of the next four years in order to minimise the total costs of road maintenance over four years. **(8 marks)**

**b** Calculate the average yearly cost if the town adopts an optimal solution. **(1 mark)**

E **5** The diagram shows the effect on a company's profits, in £1000s, of taking various advertising decisions. The company wishes to create a two-year plan that will maximise its total profit.




Each year they must decide if they will not advertise (No), advertise through television only (TV), advertise through radio only (Radio), or advertise in both media (Both).

A To determine the effectiveness of the strategy the company will estimate the value of its assets at the end of the two-year period.

Use dynamic programming to determine the advertising decisions that the company should take. **(10 marks)**

## 5.2 Minimax and maximin problems

You have already seen how to use dynamic programming to solve shortest and longest path (**minimum** and **maximum**) problems.

- **A minimax route is one in which the maximum value of the individual arcs used is as small as possible.**
- **A maximin route is one in which the minimum value of the individual arcs used is as large as possible.**

Imagine that the vertices in a network represent airports and the arcs represent distances between them. An aircraft wishes to maximise the cargo that can be carried from $S$ to $T$. Heavier cargoes require more fuel, so, assuming the aircraft can refuel at each airport, it needs a route that will minimise the length of the longest leg. This is a **minimax** problem.

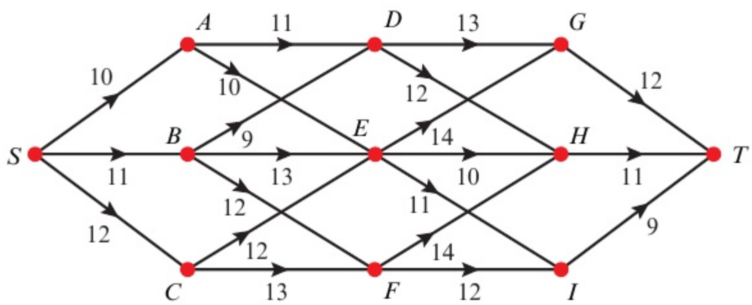
Imagine that the arcs represent the number of products processed per hour on a production line. The production rate will depend upon the slowest stage. You wish to find a route in which the smallest leg (slowest stage) is as big as possible. This would be a **maximin** problem.

**Watch out** You cannot apply Bellman's principle in the same way to minimax and maximin problems. In general, part of an optimal path from source to sink is not necessarily itself optimal.

→ Exercise 5B, Challenge

### Example 5

Use dynamic programming to find a minimax route from $S$ to $T$.




If you travel along the minimax route, the maximum arc length you meet will be as small as possible.

You are looking for the minimax route. So you need to record the maximum value of each action and select the minimum of these actions for each state.

So for **minimax**, **first** find the **maximums** and **then** the **minimum** of these.

Minimax

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| 1 | $G$ | $GT$ | $T$ | 12* |
| | $H$ | $HT$ | $T$ | 11* |
| | $I$ | $IT$ | $T$ | 9* |

| | | | | |
|---|---|---|---|---|
| 2 | $D$ | $DG$ | $G$ | Max(13, 12) = 13 |
| | | $DH$ | $H$ | Max(12, 11) = 12* |
| | $E$ | $EG$ | $G$ | Max(14, 12) = 14 |
| | | $EH$ | $H$ | Max(10, 11) = 11* |
| | | $EI$ | $I$ | Max(11, 9) = 11* |
| | $F$ | $FH$ | $H$ | Max(14, 11) = 14 |
| | | $FI$ | $I$ | Max(12, 9) = 12* |
| 3 | $A$ | $AD$ | $D$ | Max(11, 12) = 12 |
| | | $AE$ | $E$ | Max(10, 11) = 11* |
| | $B$ | $BD$ | $D$ | Max(9, 12) = 12* |
| | | $BE$ | $E$ | Max(13, 11) = 13 |
| | | $BF$ | $F$ | Max(12, 12) = 12* |
| | $C$ | $CE$ | $E$ | Max(12, 11) = 12* |
| | | $CF$ | $F$ | Max(13, 12) = 13 |
| 4 | $S$ | $SA$ | $A$ | Max(10, 11) = 11* |
| | | $SB$ | $B$ | Max(11, 12) = 12 |
| | | $SC$ | $C$ | Max(12, 12) = 12 |

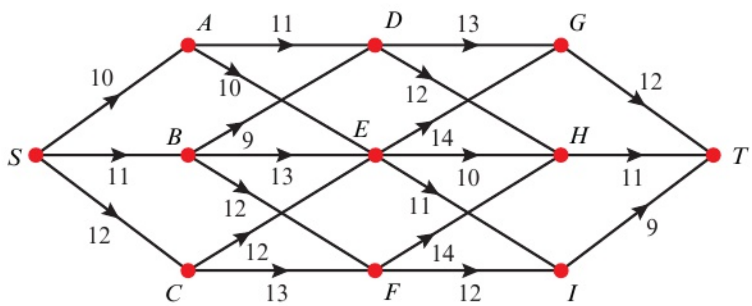
The minimax value is 11. There are two minimax routes: $SAEHT$ and $SAEIT$.

**Problem-solving**

You only need to compare two values for each action. Compare the weight of the arc being considered with the optimum value for the destination vertex from the previous stage. For example, at stage 3, the action $BD$ has weight 9, and the optimum value for vertex $D$ at stage 2 is 12. So choose the maximum of 9 and 12.

## Example 6

Use dynamic programming to find a maximin route from $S$ to $T$.




This means that, if you travel along the maximin route, the minimum arc length you meet will be as large as possible.

You are looking for the maximin route. So you need to record the minimum value of each action and select the maximum of these actions for each state.
So for **maximin**, **first** find the **minimums** and **then** the **maximum** of these.

**Maximin**

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| 1 | $G$ | $GT$ | $T$ | 12* |
| | $H$ | $HT$ | $T$ | 11* |
| | $I$ | $IT$ | $T$ | 9* |

| | | | | |
|---|---|---|---|---|
| 2 | $D$ | $DG$ | $G$ | Min(13, 12) = 12* |
| | | $DH$ | $H$ | Min(12, 11) = 11 |
| | $E$ | $EG$ | $G$ | Min(14, 12) = 12* |
| | | $EH$ | $H$ | Min(10, 11) = 10 |
| | | $EI$ | $I$ | Min(11, 9) = 9 |
| | $F$ | $FH$ | $H$ | Min(14, 11) = 11* |
| | | $FI$ | $I$ | Min(12, 9) = 9 |
| 3 | $A$ | $AD$ | $D$ | Min(11, 12) = 11* |
| | | $AE$ | $E$ | Min(10, 12) = 10 |
| | $B$ | $BD$ | $D$ | Min(9, 12) = 9 |
| | | $BE$ | $E$ | Min(13, 12) = 12* |
| | | $BF$ | $F$ | Min(12, 11) = 11 |
| | $C$ | $CE$ | $E$ | Min(12, 12) = 12* |
| | | $CF$ | $F$ | Min(13, 11) = 11 |
| 4 | $S$ | $SA$ | $A$ | Min(10, 11) = 10 |
| | | $SB$ | $B$ | Min(11, 12) = 11 |
| | | $SC$ | $C$ | Min(12, 12) = 12* |

The maximin value is 12. The maximin route is $SCEGT$.

## Exercise 5B

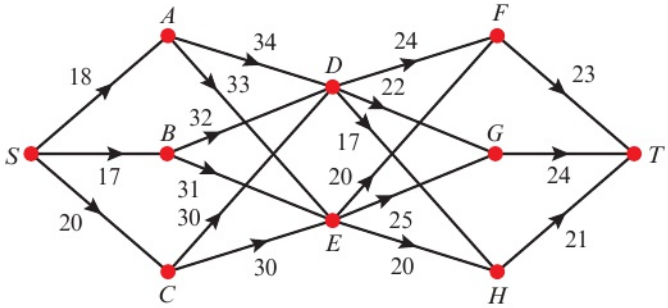
**In questions 1 to 3**, use dynamic programming to find:

**a** a minimax

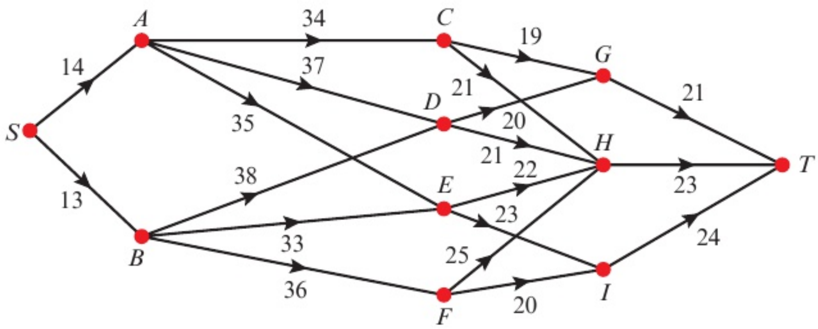
**b** a maximin

route from $S$ to $T$ in each network. State the route and its minimax or maximin value in each case.

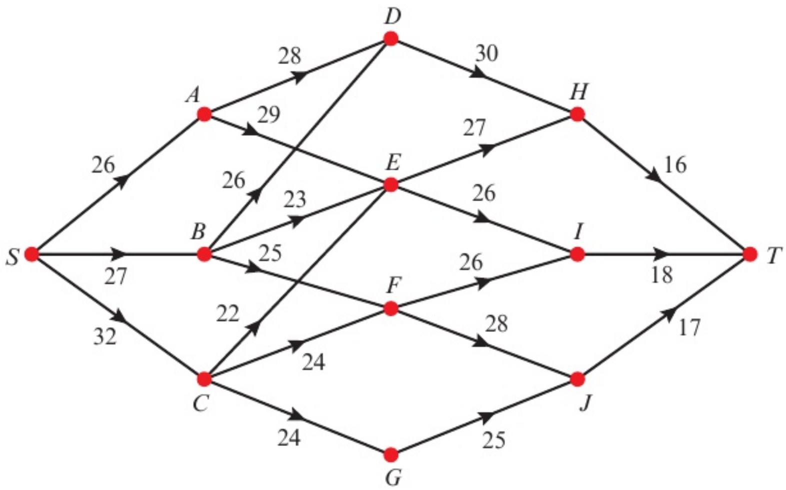
**1**



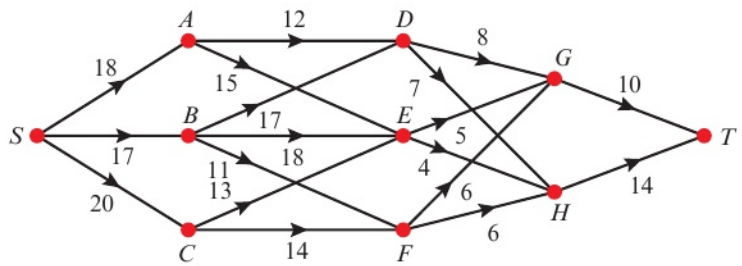
A **2**




**3**




P **4** A boating club is organising an outing from $S$ to $T$. The waterways all have different depths, and different boats have different clearances. The diagram represents the network of waterways which can only be traversed in one direction. The number on each arc represents the minimum water depth, in feet, for that waterway. The club wants to choose a route that as many boats as possible will be able to navigate.

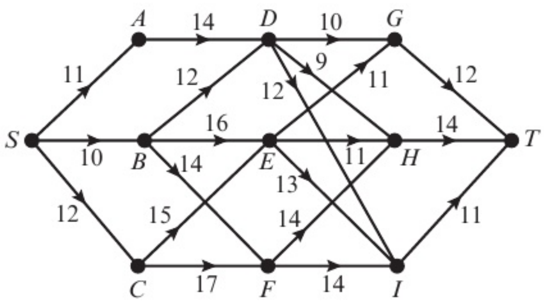



**a** State the type of problem to be solved.

**b** Use dynamic programming to choose the optimum route.

**c** State the maximum required clearance for a boat to be included in the outing.

A E/P **5** A four-day hike is planned from $S$ to $T$ based on this network. Each arc represents a day's walk and each number gives the corresponding distance, in miles.




The maximum distance covered on any day should be as small as possible.

**a** State the type of dynamic programming problem that needs to be solved. **(1 mark)**

**b** Use dynamic programming to find two possible optimal routes and state their total lengths. **(10 marks)**

**Challenge**

A student asserts that when solving a minimax problem, any part of an optimal path from source to sink is itself optimal. By drawing a suitable network, show that this is not necessarily true.

## 5.3 Dynamic programming problems in table form

You can use dynamic programming to solve optimisation problems where no network is given. Instead, information will be provided in the question text and in table form.

You will use the following terminology when finding optimal solutions to problems presented in table form:

**Stage** A time interval or an option being considered.

**State** The value of a resource available from a previous stage.

**Action** The amount of the resource used in the current stage.

**Destination** The value of the resource passed to the next stage.

**Value** The numerical value of the quantity to be optimised.

**Watch out** The terminology for problems presented in table form is the same as for problems presented in network form. However, the interpretation is different and **depends on the context of the problem**.

## Example 7

A

A fruit grower can use his surplus crop to make into jam, pie filling or dried fruit.

He can process up to five units of fruit and the expected profits, in hundreds of pounds, from making the various products are shown in the table. He will use all five units to make these products and wishes to maximise his income.

| Number of units of fruit | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Jam | 12 | 27 | 33 | 36 | 38 |
| Pie filling | 16 | 18 | 20 | 22 | 24 |
| Dried fruit | 8 | 16 | 24 | 32 | 40 |

Dynamic programming will be used to solve this problem.

**a** Define the terms Stage, State, Action, Destination and Value in this context.

**b** Determine the number of units he should allocate to each product in order to maximise his income.

**a**

| | |
|---|---|
| Stage | the product being considered: jam, pie filling, dried fruit |
| State | the number of units available |
| Action | the number of units allocated to that product |
| Destination | the number of units remaining |
| Value | the cumulative profit |

**b** The solution is presented in the form of a table.

| Stage (product) | State (units available) | Action (units allocated) | Destination (units remaining) | Value (cumulative profit) |
|---|---|---|---|---|
| 1 Dried fruit | 0 | 0 | 0 | 0* |
| | 1 | 1 | 0 | 8* |
| | 2 | 2 | 0 | 16* |
| | 3 | 3 | 0 | 24* |
| | 4 | 4 | 0 | 32* |
| | 5 | 5 | 0 | 40* |
| 2 Pie filling | 0 | 0 | 0 | 0 + 0 = 0* |
| | 1 | 1 | 0 | 16 + 0 = 16* |
| | | 0 | 1 | 0 + 8 = 8 |

**Watch out** If you are given a dynamic programming problem in word or table form, you should not, in general, attempt to draw a network to model the situation. You should use the information given to **work backwards** through the states, setting out your solution in a table.

You are working backwards, so you are considering the last product first. Since this is the last product, all the remaining surplus fruit must be used, so all the destination values must be zero. If there are $n$ units left at this stage all $n$ must be allocated to dried fruit.

Now move back a stage to the next product.

If there is one unit available then you can either allocate one unit to pie filling leaving no units for dried fruit, or no units for pie filling leaving one left for dried fruit.

A

| | | | | |
|---|---|---|---|---|
| | 2 | 2 | 0 | 18 + 0 = 18 |
| | | 1 | 1 | 16 + 8 = 24* |
| | | 0 | 2 | 0 + 16 = 16 |
| | 3 | 3 | 0 | 20 + 0 = 20 |
| | | 2 | 1 | 18 + 8 = 26 |
| | | 1 | 2 | 16 + 16 = 32* |
| | | 0 | 3 | 0 + 24 = 24 |
| | 4 | 4 | 0 | 22 + 0 = 22 |
| | | 3 | 1 | 20 + 8 = 28 |
| | | 2 | 2 | 18 + 16 = 34 |
| | | 1 | 3 | 16 + 24 = 40* |
| | | 0 | 4 | 0 + 32 = 32 |
| | 5 | 5 | 0 | 24 + 0 = 24 |
| | | 4 | 1 | 22 + 8 = 30 |
| | | 3 | 2 | 20 + 16 = 36 |
| | | 2 | 3 | 18 + 24 = 42 |
| | | 1 | 4 | 16 + 32 = 48* |
| | | 0 | 5 | 0 + 40 = 40 |
| 3 Jam | 5 | 5 | 0 | 38 + 0 = 38 |
| | | 4 | 1 | 36 + 16 = 52 |
| | | 3 | 2 | 33 + 24 = 57 |
| | | 2 | 3 | 27 + 32 = 59* |
| | | 1 | 4 | 12 + 40 = 52 |
| | | 0 | 5 | 0 + 48 = 48 |

To obtain the maximum profit 2 units should be allocated to jam, 1 to pie filling and 2 to dried fruit. This will give a profit of £5900.

If there are two units available you can allocate two, one or none to pie filling, leaving none, one or two left for dried fruit.
If you allocate 2 to pie filling the profit is 18 and you add 0 because you have no units left to allocate to dried fruit.
If you allocate 1 to pie filling the profit is 16 and you add 8 because you will use the remaining unit for dried fruit.
If you allocate no units to pie filling the profit is 0 but you add 16 because you will use the two units for dried fruit.

If there are three units available you can allocate three, two, one or none to pie filling, leaving none, one, two or three left for dried fruit.
If you allocate 3 to pie filling the profit is 20 and you add 0 because you have no units left to allocate to dried fruit.
If you allocate 2 to pie filling the profit is 18 and you add 8 because you will use the remaining units for dried fruit.
If you allocate 1 to pie filling the profit is 16 and you add 16 because you will use the remaining units for dried fruit.
If you allocate none to pie filling the profit is 0 and you add 24 because you will use the remaining units for dried fruit.

## Example 8

A clockmaker makes grandfather clocks during the winter. The order book for clocks over the next five months is shown in the table below.

| **Month** | October | November | December | January | February |
|---|---|---|---|---|---|
| **Number of clocks ordered** | 2 | 3 | 5 | 3 | 1 |

The clocks are delivered to customers at the end of each month.

He can make up to four clocks in any month, but if he needs to make more than two in any one month he will need to hire additional help at £500 per month.

The overhead costs are £300 in any month in which work is done.

He can put up to two clocks into storage at a cost of £100 per clock per month.

There are no clocks in storage at the beginning of October and there should be no clocks in storage after the February delivery.

**a** Use dynamic programming to determine the production schedule that minimises the costs.

This is an example of a production planning problem.

The clockmaker's helper needs some extra income in October, so the clockmaker decides he will definitely make four clocks this month.

**b** Given this information, use your table to determine the optimum production schedule for the remaining months, and write down the total production costs.

**Watch out** A lot of the important information for this problem is contained in the question text. Make sure you read it carefully. It can help to make a note of all the possible costs to ensure you don't forget any:
Additional help: £500 / month    Overhead: £300 / month
Storage: £100 / month per clock

**a** Stage — the month
State — the number of clocks held in storage from the previous month
Action — the number of clocks made in the month
Destination — the number of clocks put in storage at the end of the month
Value — the cumulative cost

Start by defining the terminology based on the context of the problem.

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| Feb | 1 | 0 | 0 | 100* |
| | 0 | 1 | 0 | 300* |

In February, 1 clock is required. Either 1 clock was stored in the previous month, leaving 0 to make in February, or 0 clocks were stored in the previous month so that 1 must be made in February.

| | | | | |
|---|---|---|---|---|
| Jan | 2 | 2 | 1 | 200 + 300 + 100 = 600* |
| | | 1 | 0 | 200 + 300 + 300 = 800 |
| | 1 | 3 | 1 | 100 + 300 + 500 + 100 = 1000 |
| | | 2 | 0 | 100 + 300 + 300 = 700* |
| | 0 | 4 | 1 | 0 + 300 + 500 + 100 = 900* |
| | | 3 | 0 | 0 + 300 + 500 + 300 = 1100 |

Whenever more than 2 clocks are made, an extra cost of £500 must be included.

In January, 3 clocks are required. The number stored in the previous month may be 2, 1 or 0. If 2 were stored then either 2 more are made which allows 1 to be put in storage, or just 1 is made and 0 are put in storage.

In the case where 2 are made, the cumulative costs are:

£200 (storage) + £300 (overheads) + £100 (February costs) = £600

In the case where just 1 is made, the cumulative costs are:

£200 (storage) + £300 (overheads) + £300 (February costs) = £800

| | | | | |
|---|---|---|---|---|
| Dec | 2 | 4 | 1 | 200 + 300 + 500 + 700 = 1700* |
| | | 3 | 0 | 200 + 300 + 500 + 900 = 1900 |
| | 1 | 4 | 0 | 100 + 300 + 500 + 900 = 1800* |
| Nov | 2 | 3 | 2 | 200 + 300 + 500 + 1700 = 2700 |
| | | 2 | 1 | 200 + 300 + 1800 = 2300* |
| | 1 | 4 | 2 | 100 + 300 + 500 + 1700 = 2600* |
| | | 3 | 1 | 100 + 300 + 500 + 1800 = 2700 |
| | 0 | 4 | 1 | 300 + 500 + 1800 = 2600* |
| Oct | 0 | 4 | 2 | 300 + 500 + 2300 = 3100 |
| | | 3 | 1 | 300 + 500 + 2600 = 3400 |
| | | 2 | 0 | 300 + 2600 = 2900* |

The rest of the table is completed in the same way.

So the clockmaker should make the clocks as follows.

| Month | October | November | December | January | February |
|---|---|---|---|---|---|
| Number of clocks made | 2 | 4 | 4 | 4 | 0 |

The table shows the optimal number of clocks made each month. The values are found by tracking the appropriate 'starred' items back through the table above.

The costs will be £2900.

**b** If the clockmaker makes 4 clocks in October, then the destination value for October will be 2. Tracking back through the table gives this production schedule.

| Month | October | November | December | January | February |
|---|---|---|---|---|---|
| Number of clocks made | 4 | 2 | 4 | 4 | 0 |

The total production cost is now £3100.

The total production cost is shown in the table corresponding to an action of 4 for October.

## Example 9

A builder has purchased a site and will build three houses $A$, $B$ and $C$ on it at the rate of one a year. The profit made on each house will depend on site accessibility, interest rates and cost of materials and labour. The builder's estimates of potential profits are shown in the table below, in thousands of pounds.

| Already built | *A* | *B* | *C* |
|---|---|---|---|
| None | 50 | 70 | 65 |
| *A* | – | 65 | 60 |
| *B* | 65 | – | 70 |
| *C* | 60 | 70 | – |
| *A* and *B* | – | – | 80 |
| *A* and *C* | – | 60 | – |
| *B* and *C* | 70 | – | – |

A Dynamic programming is to be used to determine the order in which the houses are to be built so that the minimum profit in each year is to be maximised.

**a** Explain the meaning of Stage, State and Action in this context.

**b** Find the order in which the houses should be built and the value of the minimum estimated annual profit.

a Stage — the number of houses left to build
State — the houses already built
Action — the house to be built

b

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| 1 | $AB$ | $C$ | $ABC$ | 80* |
| | $AC$ | $B$ | $ABC$ | 60* |
| | $BC$ | $A$ | $ABC$ | 70* |
| 2 | $A$ | $B$ | $AB$ | Min(65, 80) = 65* |
| | | $C$ | $AC$ | Min(60, 60) = 60 |
| | $B$ | $A$ | $AB$ | Min(65, 80) = 65 |
| | | $C$ | $BC$ | Min(70, 70) = 70* |
| | $C$ | $A$ | $AC$ | Min(60, 60) = 60 |
| | | $B$ | $BC$ | Min(70, 70) = 70* |
| 3 | None | $A$ | $A$ | Min(50, 65) = 50 |
| | | $B$ | $B$ | Min(70, 70) = 70* |
| | | $C$ | $C$ | Min(65, 70) = 65 |

The houses should be built in the order $BCA$. The minimum estimated annual profit is £70 000.

**Problem-solving**

By defining the stage as the number of houses **left to build**, your stages will be numbered in the order that you need to consider them.

You need to maximise the least profit, so this is a maximin problem.

## Example 10

Toby has a fairground ride and will take it to three county fairs over the next three weeks.

He will leave home at the start of week 1 and travel to the first fair, then go to the second fair for the second week and then go to the third fair for the final week before returning home.

There are seven fairs during the next three weeks so there is a choice of fair each week. Toby must decide which one of the fairs he should go to.

Table 1 gives the week in which each fair is held. Table 2 gives the expected profits, in hundreds of pounds, in going to each fair. Table 3 gives the travel costs in hundreds of pounds.

Table 1

| **Week** | 1 | 2 | 3 |
|---|---|---|---|
| **Fair** | $A$, $B$ | $C$, $D$, $E$ | $F$, $G$ |

Table 2

| **Fair** | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ |
|---|---|---|---|---|---|---|---|
| **Profit (£100s)** | 6 | 5 | 8 | 10 | 11 | 14 | 12 |

A Table 3

| Travel costs (£100s) | *A* | *B* | *C* | *D* | *E* | *F* | *G* |
|---|---|---|---|---|---|---|---|
| Home | 3 | 2 | | | | 8 | 7 |
| *A* | | | 3 | 2 | 5 | | |
| *B* | | | 1 | 3 | 3 | | |
| *C* | | | | | | 5 | 5 |
| *D* | | | | | | 4 | 2 |
| *E* | | | | | | 4 | 1 |

Toby wishes to maximise his income.

Use dynamic programming to determine the fairs he should go to and his total income.

Stage — the number of weeks remaining
State — the fair Toby is currently attending
Action — the journey that Toby will make next
Destination — the fair Toby will attend next

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| 1 | *F* | *F*–Home | Home | 14 − 8 = 6* |
| | *G* | *G*–Home | Home | 12 − 7 = 5* |
| 2 | *C* | *CF* | *F* | 8 − 5 + 6 = 9* |
| | | *CG* | *G* | 8 − 5 + 5 = 8 |
| | *D* | *DF* | *F* | 10 − 4 + 6 = 12 |
| | | *DG* | *G* | 10 − 2 + 5 = 13* |
| | *E* | *EF* | *F* | 11 − 4 + 6 = 13 |
| | | *EG* | *G* | 11 − 1 + 5 = 15* |
| 3 | *A* | *AC* | *C* | 6 − 3 + 9 = 12 |
| | | *AD* | *D* | 6 − 2 + 13 = 17* |
| | | *AE* | *E* | 6 − 5 + 15 = 16 |
| | *B* | *BC* | *C* | 5 − 1 + 9 = 13 |
| | | *BD* | *D* | 5 − 3 + 13 = 15 |
| | | *BE* | *E* | 5 − 3 + 15 = 17* |
| 4 | Home | Home–*A* | *A* | −3 + 17 = 14 |
| | | Home–*B* | *B* | −2 + 17 = 15* |

Toby should attend fairs *B*, *E* and *G*. He will make an income of £1500.

Start by defining the terminology based on the context of the problem. At each stage, the value will be determined by subtracting the cost of the journey from the profit made. You want to maximise the total income, so this is a maximisation problem.

The value here is made up of the profit from attending *E*, minus the travel cost from *E* to *F*, plus the value found from state *F*.

A

## Exercise 5C

E/P 1 A company is created to sell holidays on an island. There are three new resorts $A$, $B$ and $C$ being created on the island and the company decides to introduce one new resort to its catalogue each year over the next three years. The estimates of annual costs are shown in the table below, in hundreds of pounds.

| Resorts listed | $A$ | $B$ | $C$ |
|---|---|---|---|
| None | 55 | 70 | 60 |
| $A$ | – | 65 | 75 |
| $B$ | 70 | – | 65 |
| $C$ | 75 | 80 | – |
| $A$ and $B$ | – | – | 75 |
| $A$ and $C$ | – | 85 | – |
| $B$ and $C$ | 60 | – | – |

For funding reasons, the company needs to choose the order in which the resorts are introduced so that the greatest annual cost is as small as possible.

Dynamic programming will be used to determine the order in which the resorts are introduced.

**a** State the type of dynamic programming problem to be solved. **(1 mark)**

**b** Explain the meaning of Stage, State and Action in this context. **(3 marks)**

**c** Copy and complete the table. **(6 marks)**

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| 1 | $BC$ | $A$ | $ABC$ | 60* |
| | $AC$ | $B$ | $ABC$ | 85* |
| | $AB$ | $C$ | $ABC$ | 75* |
| 2 | $A$ | $B$ | $AB$ | Max(65, 75) = 75 |
| | | | | |

**d** Find the order in which the resorts should be added and the greatest annual cost. **(2 marks)**

E/P 2 A house renovation project is to be completed in six weeks (30 working days). The work is in four stages: clearance, repairing, modernisation and decorating, which must be undertaken in that order. The cost, in £1000s, of each stage depends on the time taken to do it. These costs are shown in the table.

| Time for stage (days) | Clearance | Repairing | Modernisation | Decorating |
|---|---|---|---|---|
| 5 | 15 | 24 | 22 | 14 |
| 10 | 13 | 20 | 19 | 12 |
| 15 | 8 | 15 | 15 | 9 |
| 20 | 5 | 10 | 11 | 4 |
| 25 | 2 | 6 | 7 | 2 |

A Dynamic programming will be used to solve this problem.

a Define the terms Stage, State, Action, Destination and Value in this context. **(4 marks)**

b Determine the number of days that should be allocated to each stage in order to minimise costs. **(12 marks)**

E/P **3** A company makes aircraft. The order book over the next four months is shown in the table below.

| **Month** | March | April | May | June |
|---|---|---|---|---|
| **Number of aircraft ordered** | 1 | 2 | 3 | 2 |

The aircraft are delivered to customers at the end of each month.

Up to three aircraft can be made in any month, but if more than two are made in any one month additional equipment will need to be hired at a cost of £20 000 per month.

If any work is done in a month the overhead costs are £50 000.

Up to two aircraft can be held in secure hangers at a cost of £10 000 per aircraft per month.

There are no aircraft in store at the beginning of March and there should be no aircraft in store after the June delivery.

a Use dynamic programming to determine the production schedule that minimises the costs. **(12 marks)**

b State Bellman's principle of optimality. **(1 mark)**

Due to staff shortages, a maximum of one aircraft can be made in March.

c Given this information, use your table to determine the production schedule that minimises costs, and write down the total production costs. **(2 marks)**

E/P **4** A salesman will visit four shops in the next four days to demonstrate a new product. He will start at home and travel to the first shop and spend the first day there, then travel directly to the second shop for day 2, on to the third shop for day 3, then to the fourth shop for day 4, and then travel home.

Table 1 shows the shops he could visit on each day.
Table 2 shows the anticipated profits, in £100s, from sales at each shop.
Table 3 shows the travelling expenses, in £100s, that will be incurred.

The company employing the salesman wishes to maximise the income, after subtracting the travel costs, generated by the salesman's visits. Find his optimum route.

Table 1

| **Monday** | **Tuesday** | **Wednesday** | **Thursday** |
|---|---|---|---|
| *A, B, C* | *D, E* | *F, G* | *H, I, J* |

Table 2

| **Shop** | *A* | *B* | *C* | *D* | *E* | *F* | *G* | *H* | *I* | *J* |
|---|---|---|---|---|---|---|---|---|---|---|
| **Profit** | 8 | 9 | 8 | 12 | 14 | 10 | 11 | 14 | 13 | 11 |

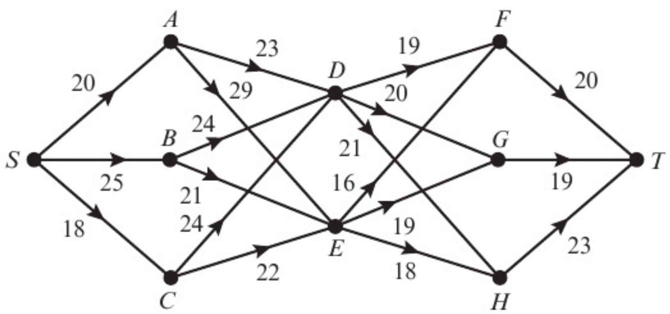
A Table 3

| | *A* | *B* | *C* | *D* | *E* | *F* | *G* | *H* | *I* | *J* |
|---|---|---|---|---|---|---|---|---|---|---|
| **Home** | 2 | 2 | 3 | | | | | 6 | 4 | 3 |
| ***A*** | | | | 3 | 4 | | | | | |
| ***B*** | | | | 4 | 6 | | | | | |
| ***C*** | | | | 4 | 4 | | | | | |
| ***D*** | | | | | | 5 | 5 | | | |
| ***E*** | | | | | | 4 | 7 | | | |
| ***F*** | | | | | | | | 5 | 4 | 4 |
| ***G*** | | | | | | | | 5 | 5 | 4 |

**(12 marks)**

## Mixed exercise 5

E **1**




**a** Use dynamic programming to find the minimax route from $S$ to $T$ in the network above. **(8 marks)**

**b** Give the minimax route from $C$ to $T$. **(1 mark)**

**2** Using the same diagram as in question 1, use dynamic programming to find the maximin route.

E/P **3** A dairy manufacturer can make butter, cheese and yoghurt. Up to five units of milk can be processed and the profits, in pounds, from the various allocations are shown in the table.

| **Number of units** | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **Butter** | 14 | 25 | 34 | 41 | 47 |
| **Cheese** | 12 | 30 | 40 | 45 | 49 |
| **Yoghurt** | 10 | 20 | 30 | 40 | 50 |

The manufacturer wishes to maximise his profit.

**a** Use dynamic programming to find an optimal solution and state the profit. **(8 marks)**

**b** Show that there is a second optimal solution. **(2 marks)**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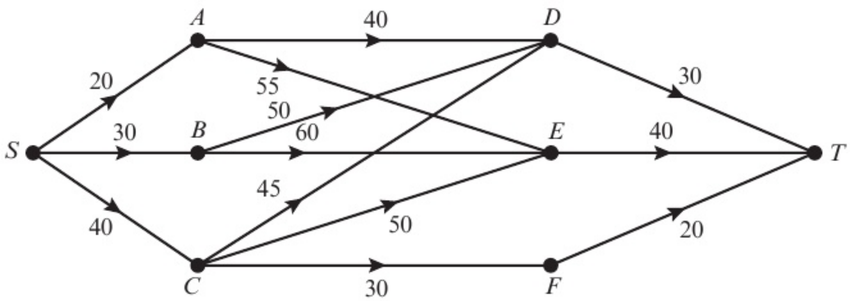
A E/P 4 The diagram shows the possible routes from $S$ to $T$. The weights on the arcs give the maximum altitude on the road (in units of 100 feet). Jenny wishes to travel from $S$ to $T$. She wishes to choose the route on which the maximum altitude, above sea level, is as small as possible. This is called the minimax route.




Use dynamic programming, carefully defining the stages and states, to determine the route or routes Jenny should take. You should show your calculations in tabular form. **(10 marks)**

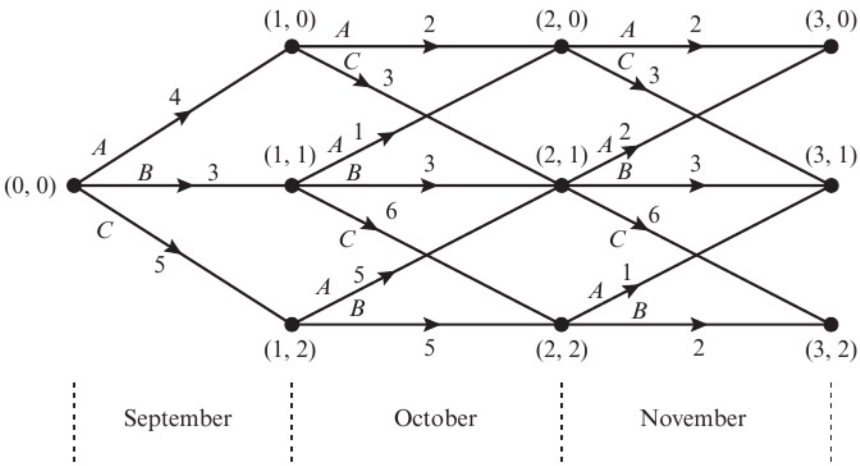
E/P 5 At the beginning of each month an advertising manager must choose one of three adverts:

- the previous advert
- the current advert
- a new advert

She therefore has three options:

- $A$ use the previous advert
- $B$ use the current advert
- $C$ run a new advert

The possible choices are shown in the network below, together with (stage, state) variables at the vertices and the expected profits, in thousands of pounds, on the arcs.




The manager wants to maximise her profits for the three-month period.

**a** Use dynamic programming to determine the decisions the manager should take. Show your calculations in tabular form. **(7 marks)**

**b** Hence obtain the sequence of decisions she should make to obtain the maximum profit. State the maximum profit. **(3 marks)**

A E/P **6** A small company builds car trailers.

The overhead costs for building up to 4 trailers in a month is £1250. Building more than 4 trailers in any month costs an extra £300. The maximum number of trailers that can be built in a month is 6.

Up to 2 trailers may be stored from one month to the next at a cost of £100 per trailer.

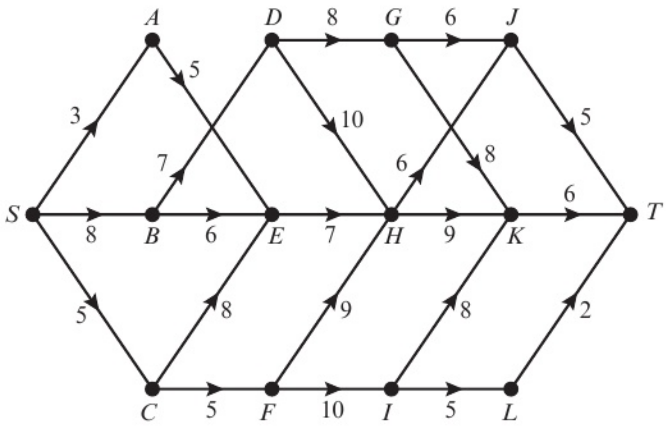
Each trailer is either delivered, or stored, at the end of a month. At the start of April, there are no trailers stored. There must be no trailers stored at the end of July.

The order book for trailers is

| **Month** | April | May | June | July |
|---|---|---|---|---|
| **Number ordered** | 3 | 7 | 5 | 3 |

Use dynamic programming to determine the schedule that minimises the production costs. State the minimum cost to complete the orders. **(12 marks)**

E/P **7** The staged, directed network represents a course used in a kayaking competition. Each competitor must start at $S$, pass through each stage and finish at $T$.




Sam has graded each section linking one stage to the next, according to the level of difficulty, where 10 is the hardest. She wants to find a route where the highest number is minimised.

**a** Write down the type of dynamic programming problem that Sam needs to solve. **(1 mark)**

**b** Use dynamic programming to solve the problem, stating the route she should use. **(10 marks)**

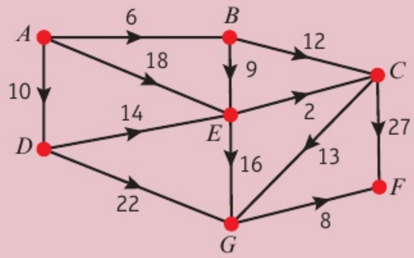
Sam practises section $FH$ of the course until she reassesses its difficulty level to 6.

**c** State the route that Sam should now take. **(1 mark)**

## Challenge

A

**1** Use Dijkstra's algorithm to find the shortest path from $A$ to $F$ in this network.




Explain the complication in attempting to solve the same problem using dynamic programming.

**2** Look at the production problem in question **6** in Mixed exercise 5.
Formulate this problem as a linear programming problem.

**Problem-solving**

Depending on how you define your decision variables, you might need to make use of the **floor** and **ceiling** functions in your objective function:
$\lfloor x \rfloor$ is the largest integer less than or equal to $x$
$\lceil x \rceil$ is the smallest integer greater than or equal to $x$.

## Summary of key points

A

**1** Any part of the shortest/longest path from $S$ to $T$ is itself a shortest/longest path. This is known as **Bellman's principle of optimality** and is sometimes stated as 'any part of an optimal path is optimal'.

**2** Use the following terminology for **dynamic programming** problems presented in **network form**:

| | |
|---|---|
| **Stage** | The route from the initial state ($S$ say) to the final state ($T$ say) is made up of a sequence of moves. Each move is a stage. The stage tells how 'far' you are from the destination vertex. |
| **State** | This is the vertex that you are considering at each point. |
| **Action** | This is the directed arc from one state to the next. In selecting an arc you are considering what happens if you do that action. |
| **Destination** | This is the vertex you arrive at having taken the action. |
| **Value** | This is the sum of the weights on the arcs used in a sequence of actions. |

**3** In **dynamic programming** you **work backwards from $T$** in a series of stages.

- The vertices immediately before $T$ are examined. These are the stage 1 vertices. The best route from these to $T$ is noted.
- Then move back a stage, away from $T$ and towards $S$, to the stage 2 vertices. Routes from these vertices must pass through one of the stage 1 vertices to get to $T$. Use this to help you find the optimal route from each stage 2 vertex to $T$.
- Then move back a stage and repeat the process, until you reach $S$. The principle of optimality is used at each stage. The current optimal paths are developed using the paths found in the previous stage.

**4**
- A **minimax** route is one in which the maximum value of the individual arcs used is as small as possible.
- A **maximin** route is one in which the minimum value of the individual arcs used is as large as possible.

**5** Use the following terminology for **dynamic programming** problems presented in **table form**:

| | |
|---|---|
| **Stage** | A time interval or an option being considered. |
| **State** | The value of a resource available from a previous stage. |
| **Action** | The amount of the resource used in the current stage. |
| **Destination** | The value of the resource passed to the next stage. |
| **Value** | The numerical value of the quantity to be optimised. |

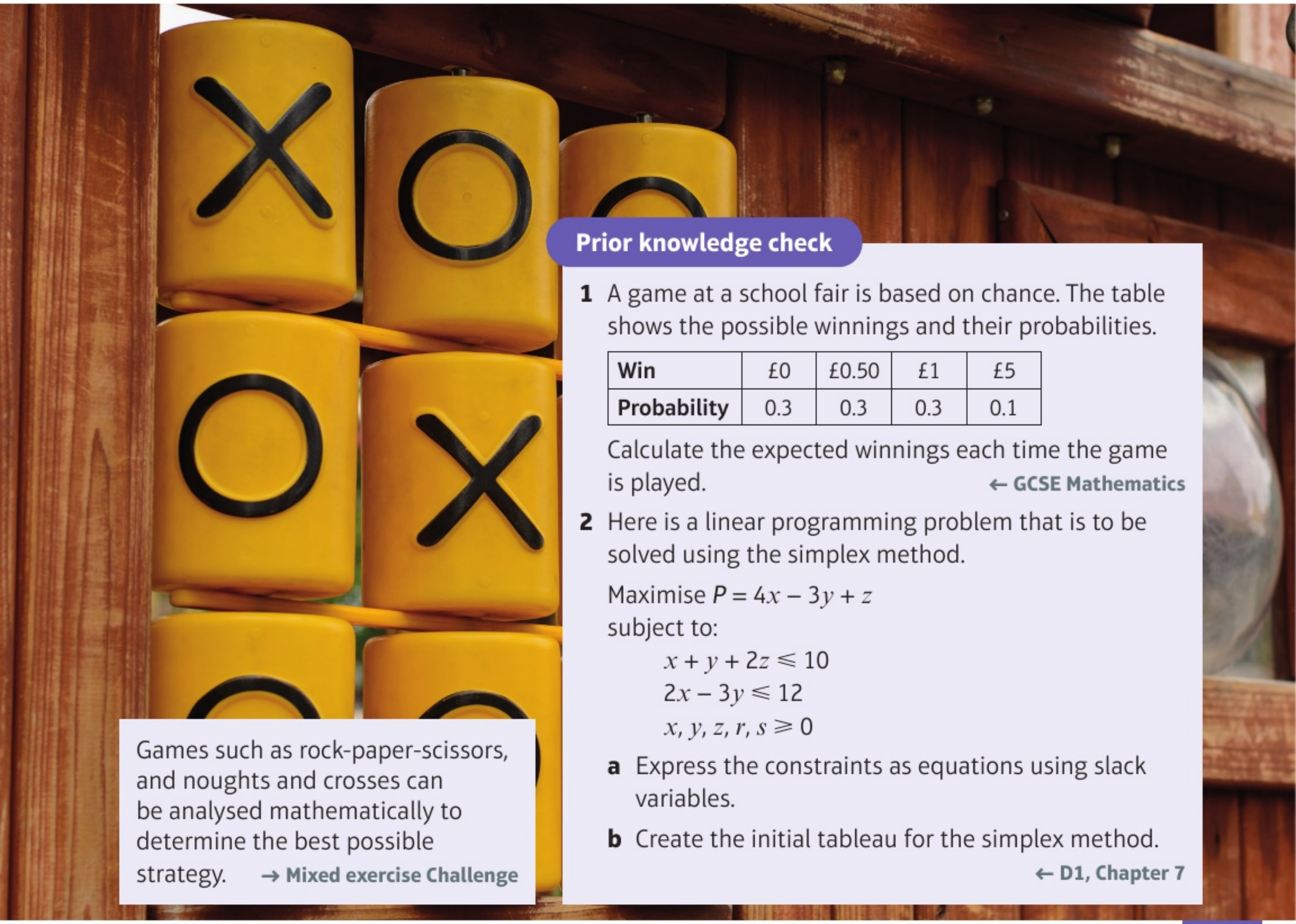
# Game theory

# 6

## Objectives

After completing this chapter you should be able to:

- Understand two-person games and the pay-off matrix → pages 180–213
- Determine play-safe strategies and stable solutions (saddle points) → pages 180–191
- Reduce a pay-off matrix using dominance arguments → pages 191–194
- Determine the optimal mixed strategy for a game with no stable solution, for the player with two choices in a 2 × 3, 3 × 2, 2 × 4 or 4 × 2 game → pages 194–203
- Convert games into linear programming problems → pages 203–209



Games such as rock-paper-scissors, and noughts and crosses can be analysed mathematically to determine the best possible strategy. → Mixed exercise Challenge

## Prior knowledge check

**1** A game at a school fair is based on chance. The table shows the possible winnings and their probabilities.

| **Win** | £0 | £0.50 | £1 | £5 |
|---|---|---|---|---|
| **Probability** | 0.3 | 0.3 | 0.3 | 0.1 |

Calculate the expected winnings each time the game is played. ← GCSE Mathematics

**2** Here is a linear programming problem that is to be solved using the simplex method.

Maximise $P = 4x - 3y + z$

subject to:

$$x + y + 2z \leqslant 10$$
$$2x - 3y \leqslant 12$$
$$x, y, z, r, s \geqslant 0$$

**a** Express the constraints as equations using slack variables.

**b** Create the initial tableau for the simplex method.

← D1, Chapter 7

## 6.1 Play-safe strategies and stable solutions

A well-known classic example of game theory is the Prisoner's Dilemma which, in one version, goes as follows.

> Two men are caught trying to spend a large number of forged £50 notes and are arrested by the police. The inspector in charge of the investigation is convinced that these two men are not only guilty of trying to spend forged notes but are also the counterfeiters. He has no evidence that will stand up in court at present, so he puts the men in different rooms and makes the same proposition **separately** to each of the arrested men.
>
> 'If **neither** of you confesses to being a counterfeiter, then we will charge both of you with attempting to pass forged notes – I expect you will get about one year for that crime.
>
> Should **both** of you confess to being forgers, than we will do our best to get a lenient sentence – I would expect about four years.
>
> However, if **only** you confess to forgery, then we will get you a free pardon, but we will throw the book at your fellow prisoner – and I expect he will get about 10 years.'

This is an example of a two-person game.

- **A two-person game is one in which only two parties can play.**

Label the two prisoners $A$ and $B$. Prisoner $A$ has two possible choices:

- Choice 1: do not confess
- Choice 2: confess

**Choice 1:** If prisoner $A$ chooses not to confess, there are two possible outcomes. If prisoner $B$ does not confess, prisoner $A$ receives a one-year sentence. But if prisoner $B$ does confess, prisoner $A$ can expect to get a 10-year sentence.

So the best possible outcome is one year, and the worst possible outcome is 10 years.

**Choice 2:** If prisoner $A$ confesses, then there are also two possible outcomes. If prisoner $B$ does not confess, prisoner $A$ gets a free pardon, but if prisoner $B$ confesses, prisoner $A$ can expect to get a four-year sentence.

So the best possible outcome is freedom (0 years) and the worst possible outcome is four years.

You can summarise this information in a table. This table is called a **pay-off matrix**.

| | ***B* confesses** | ***B* does not confess** |
|---|---|---|
| ***A* confesses** | (−4, −4) | (0, −10) |
| ***A* does not confess** | (−10, 0) | (−1, −1) |

The worst possible outcome for $A$ with this choice is −4 (four years in prison).

The worst possible outcome for $A$ with this choice is −10 (10 years in prison).

If $A$ wants to **play safe** then he should consider the worst possible outcome for each choice, and choose the better of these. In this case, the least worst possible outcome is −4, so $A$ should choose to confess.

If $B$ also wishes to play safe, then he should also choose to confess.

If both players adopt this strategy, then they will both confess, which will mean they each get four years (and, so the story goes, the inspector gets a promotion).

- **When playing safe each player looks for the worst that could happen if they make each choice in turn. The player then picks the choice that results in the least worst option.**

If a player adopts a play-safe strategy, then they will play the same choice every time they play the game. A play-safe strategy is an example of a **pure strategy**.

- **A strategy in which a player always makes the same choice is called a pure strategy.**

**Watch out** Player $A$ can improve his outcome by choosing to confess, regardless of the choice player $B$ makes. (Either from −10 to −4, or from −1 to 0.) The same argument is true for player $B$, so if both players use this optimal strategy, they will each get 4 years. This is not the outcome which is mutually best for both players, which is why the situation is referred to as the Prisoner's Dilemma.

**Links** Strategies that involve varying your choice each time you play are called **mixed strategies**. → Section 6.3

## Example 1

In this pay-off matrix player $A$ has a choice of four options and player $B$ has a choice of three options. The outcomes are given as ordered pairs, ($A$'s winnings, $B$'s winnings). Determine the play-safe strategy for each player.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | (8, 2) | (0, 9) | (7, 3) |
| ***A* plays 2** | (3, 6) | (9, 0) | (2, 7) |
| ***A* plays 3** | (1, 7) | (6, 4) | (8, 1) |
| ***A* plays 4** | (4, 2) | (4, 6) | (5, 1) |

**Notation** If $A$ chooses option 1 you say that '$A$ plays 1'.

Record the worst outcomes for each decision for each player.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** | Worst outcome for $A$ |
|---|---|---|---|---|
| ***A* plays 1** | (8, 2) | (0, 9) | (7, 3) | 0 |
| ***A* plays 2** | (3, 6) | (9, 0) | (2, 7) | 2 |
| ***A* plays 3** | (1, 7) | (6, 4) | (8, 1) | 1 |
| ***A* plays 4** | (4, 2) | (4, 6) | (5, 1) | 4 |
| Worst outcome for $B$ | 2 | 0 | 1 | |

If $B$ plays 1, then $B$ can win 2, 6, 7 or 2, depending on whether $A$ plays 1, 2, 3 or 4. So the worst outcome for this option is 2.

Look at $A$'s worst outcomes and choose the option that gives $A$ the best result. In this case $A$ should play 4.

Look at $B$'s worst outcomes. $B$ should play 1.

If $A$ and $B$ both play safe, the outcome is (4, 2). $A$ wins 4 and $B$ wins 2.

**Problem-solving**

In the above example both players could win more by **collaborating**. For example, if $A$ plays 3 and $B$ plays 2, the result is (6, 4) and both players increase their winnings.

Also, either player could increase their winnings if they know the other player is planning to play safe. For example, if $A$ knows that $B$ will play 1, then $A$ can win 8 by also playing 1. Similarly, if $B$ knows that $A$ will play 4, then $B$ can win 6 by playing 2.

In the example above, the amount won by both players is greater than or equal to 0 for every outcome, and it is assumed that a win by one player does not result in a corresponding loss for the other. You can think of the winnings as being paid by some imaginary external banker.

In this chapter, you will consider situations where there is no external banker, so that any winnings by one player are paid out by the other player. Games such as this are referred to as **zero-sum games**.

If you use a negative number to indicate a loss, you can write a pay-off matrix for a zero-sum game. Each entry will be in the form $(k, -k)$, since any amount won by player $A$ means that player $B$ loses a corresponding amount, and vice versa.

| | $B$ plays 1 | $B$ plays 2 | $B$ plays 3 |
|---|---|---|---|
| $A$ plays 1 | (3, −3) | (−4, 4) | (2, −2) |
| $A$ plays 2 | (−1, 1) | (4, −4) | (−2, 2) |
| $A$ plays 3 | (−3, 3) | (1, −1) | (4, −4) |
| $A$ plays 4 | (1, −1) | (−1, 1) | (1, −1) |

**Note** Collaboration is not beneficial in a zero-sum game – the more $A$ wins, the more $B$ loses, and vice versa.

- **In a zero-sum game the two entries in each cell in the pay-off matrix add up to zero.**

This fact means that you can represent the pay-off matrix for a zero-sum game using only **one value** in each position. Usually, the value in each position represents **player $A$'s winnings**. You can find player $B$'s winnings for the same entry by multiplying that value by −1.

The above pay-off matrix would be written as:

| | $B$ plays 1 | $B$ plays 2 | $B$ plays 3 |
|---|---|---|---|
| $A$ plays 1 | 3 | −4 | 2 |
| $A$ plays 2 | −1 | 4 | −2 |
| $A$ plays 3 | −3 | 1 | 4 |
| $A$ plays 4 | 1 | −1 | 1 |

Remember that these numbers represent $A$'s winnings and that $B$'s winnings are the negative of each number in the table.

- **A pay-off matrix is always written from the row player's ($A$'s) point of view unless you are told otherwise.**

You need to be able to find the **play-safe strategy** for each player in a zero-sum game. This is the strategy that maximises the minimum pay-off for the player. In the case of player $A$ this will mean selecting the row with the largest possible minimum pay-off. For player $B$ this will mean selecting the column with the lowest possible maximum pay-off. This is because the entries represent $A$'s winnings, so they also represent $B$'s losses.

- **The play-safe strategies are:**
  - **For player $A$, always play the row containing the row maximin.**
  - **For player $B$, always play the column containing the column minimax.**

**Notation** The **row maximin** is the maximum of the minimum values in each row. Find the minimum value for each row and choose the largest of these. The **column minimax** is the minimum of the maximum values in each column. Find the maximum value for each column and choose the smallest of these.

## Example 2

Determine the play-safe strategy for both players, $A$ and $B$, for the game below.

| | $B$ plays 1 | $B$ plays 2 | $B$ plays 3 |
|---|---|---|---|
| $A$ plays 1 | 3 | –4 | 2 |
| $A$ plays 2 | –1 | 4 | –2 |
| $A$ plays 3 | –3 | 1 | 4 |
| $A$ plays 4 | 1 | –1 | 1 |

Look for player $A$'s play-safe strategy.
Record the minimum number in each row and choose the row containing the maximum of these minimums.

| | $B$ plays 1 | $B$ plays 2 | $B$ plays 3 | Row minimum |
|---|---|---|---|---|
| $A$ plays 1 | 3 | –4 | 3 | –4 |
| $A$ plays 2 | –1 | 4 | –2 | –2 |
| $A$ plays 3 | –3 | 1 | 4 | –3 |
| $A$ plays 4 | 1 | –1 | 1 | –1 |

The maximum of these minimums (**maximin**) is –1, which appears in row 4, so $A$ plays 4.
Look for player $B$'s play-safe strategy.
Record the maximum number in each column and choose the column containing the minimum of these maximums.

| | $B$ plays 1 | $B$ plays 2 | $B$ plays 3 |
|---|---|---|---|
| $A$ plays 1 | 3 | –4 | 3 |
| $A$ plays 2 | –1 | 4 | –2 |
| $A$ plays 3 | –3 | 1 | 4 |
| $A$ plays 4 | 1 | –1 | 1 |
| Column maximum | 3 | 4 | 4 |

The minimum of these maximums (**minimax**) is 3, which appears in column 1, so $B$ plays 1.

Remember that the pay-off matrix shows the game from $A$'s point of view. So a 3 means –3 from $B$'s point of view. $B$'s worst outcome in each column will be given by the biggest number, since this is $A$'s biggest winning number and therefore $B$'s biggest loss, and you want the smallest of these.

In the above example, if either player knows that the other player will adopt a play-safe strategy, then they can increase their winnings by choosing a different strategy. You can see this by looking at the row and column corresponding to the play-safe strategies:

| | $B$ plays 1 | $B$ plays 2 | $B$ plays 3 |
|---|---|---|---|
| $A$ plays 1 | 3 | –4 | 3 |
| $A$ plays 2 | –1 | 4 | –2 |
| $A$ plays 3 | –3 | 1 | 4 |
| $A$ plays 4 | 1 | –1 | 1 |

If $B$ plays safe (by playing 1), $A$ can win 3 by playing 1.

If $A$ plays safe (by playing 4), $B$ can win 1 by playing 2.

When both players play safe, $A$ wins 1 and $B$ loses 1.

In some cases it is possible to find pure strategies for both players in which neither player can improve their winnings in this way. Such games are said to have a **stable solution**.

**Notation** If both players in a game have chosen strategies, and neither player can improve their outcome by unilaterally choosing a different strategy, then the game is said to be in **equilibrium**.

- **A two-person zero-sum game has a stable solution if there exist pure strategies for both players in which neither player has an incentive to change their strategy.**

The following example shows a two-person zero-sum game with a stable solution.

## Example 3

Anna and Ben play a two-person zero-sum game, which is represented by the following pay-off matrix.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 | *B* plays 4 |
|---|---|---|---|---|
| *A* plays 1 | 4 | −1 | 2 | 3 |
| *A* plays 2 | 4 | 6 | 3 | 7 |
| *A* plays 3 | 1 | 2 | −2 | 4 |

Find the play-safe strategy for each player and show that the game has a stable solution.

Look for the play-safe strategies.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 | *B* plays 4 | Row minimum |
|---|---|---|---|---|---|
| *A* plays 1 | 4 | −1 | 2 | 3 | −1 |
| *A* plays 2 | 4 | 6 | 3 | 7 | 3 |
| *A* plays 3 | 1 | 2 | −2 | 4 | −2 |
| Column maximum | 4 | 6 | 3 | 7 | |

The row maximin is 3, so *A* should play 2.
The column minimax is 3, so *B* should play 3.
If both players adopt a play-safe strategy, *A* will win 3.

Consider *A*'s reasoning.
*A* assumes that *B* will play column 3.
If *A* plays row 1 she only wins 2 and if she plays row 3 she will lose 2.
So there is no incentive for *A* to change her strategy.

You now need to decide if it is worth *A* or *B* changing their strategy.

Consider *B*'s reasoning.
*B* assumes that *A* will play row 2.
If *B* plays column 1 he loses 4, column 2 he loses 6 and column 4 he loses 7. All of these are worse than the current situation where *B* is losing 3.
So there is no incentive for *B* to change his strategy.

Since neither player has any incentive to change from their play-safe strategy, the game has a stable solution.

When both players adopt their play-safe strategies, the game is in equilibrium.

If you look at the row and column corresponding to the play-safe strategies in the above example, you can see that the outcome of the game represents the **smallest value in its row** and the **largest value in its column**. Such a value is called a **saddle point**.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 | *B* plays 4 |
|---|---|---|---|---|
| *A* plays 1 | 4 | −1 | 2 | 3 |
| *A* plays 2 | 4 | 6 | 3 | 7 |
| *A* plays 3 | 1 | 2 | −2 | 4 |

3 is smaller than 4, 6 or 7, but larger than 2 or −2, so it is a saddle point.

- **A saddle point in a pay-off matrix is a value which is the smallest in its row and the largest in its column.**

**Watch out** A game can have more than one saddle point, but they must all have the same value. → Exercise 6A Challenge

The reasoning in the example above shows that saddle points correspond directly with stable solutions in two-person zero-sum games.

- **A two-person zero-sum game has a stable solution if and only if its pay-off matrix contains a saddle point.**

**Notation**
The **value** (to player *A*) of a game with a stable solution is the value at the saddle point. For the game in Example 3:
- value to player $A = 3$
- value to player $B = -3$

In practice, it is inefficient to find saddle points in a pay-off matrix by inspection. You should use the following theorem to determine whether a game has a stable solution.

- **In a two-person zero-sum game there will be a stable solution if and only if**
  **the row maximin = the column minimax**

## Proof of the stable solution theorem

**Note** The proof is not needed for the examination.

Start by defining V($A$) and V($B$) as follows:
V($A$) = row maximin
V($B$) = −column minimax element
Then prove the following two theorems:

**Theorem 1: For any zero-sum game V($A$) + V($B$) $\leqslant$ 0**

If both players play safe, $A$ wins at least V($A$) and $B$ wins at least V($B$) so the total winnings are at least V($A$) + V($B$). However, in a zero-sum game the total winnings are always zero. Hence V($A$) + V($B$) cannot exceed zero.

**Theorem 2: A zero-sum game has a stable solution if and only if V($A$) + V($B$) = 0**

**The 'if' bit** (If V($A$) + V($B$) = 0 then the game has a stable solution):

Since V($A$) is the row maximin, there is a row, say the $r$th, in which the smallest element is V($A$). ($A$ will therefore be playing safe by playing row $r$.)

Similarly since V($B$) is the column minimax, there is a column, say the $s$th, in which the largest entry is −V($B$). ($B$ will therefore be playing safe by playing column $s$.)

**Notation** 'If and only if' means that each statement must imply the other. You need to prove that if V($A$) + V($B$) = 0 then there is a stable solution, **and** that if there is a stable solution, then V($A$) + V($B$) = 0.

Consider the element $x_{r,s}$ in the pay-off matrix where row $r$ intersects column $s$.

Now you know that $(x_{r,s}) \geqslant \text{V}(A)$ since $\text{V}(A)$ is the smallest number in row $r$.

Also you know that $(x_{r,s}) \leqslant -\text{V}(B)$ since $-\text{V}(B)$ is the biggest number in column $s$.

Thus

$$\text{V}(A) \leqslant (x_{r,s}) \leqslant -\text{V}(B)$$

but you are assuming that $\text{V}(A) + \text{V}(B) = 0$, hence $\text{V}(A) = -\text{V}(B)$, hence

$$\text{V}(A) \leqslant (x_{r,s}) \leqslant \text{V}(A)$$

Hence $(x_{r,s}) = \text{V}(A) = -\text{V}(B)$

This means that $(x_{r,s})$ is a saddle point and the game has a stable solution.

**The 'only if' bit** (If there is stable solution then $\text{V}(A) + \text{V}(B) = 0$):

If the game has a stable solution, then there is a saddle point (or points). Say that a saddle point occurs in row $r$ and column $s$, and has the value $(x_{r,s})$.

Since $(x_{r,s})$ is the least value in its row:

$$(x_{r,s}) \leqslant \text{V}(A)$$

Since $(x_{r,s})$ is the greatest value in its column:

$$(x_{r,s}) \geqslant -\text{V}(B)$$

Hence $-\text{V}(B) \leqslant (x_{r,s}) \leqslant \text{V}(A)$, and $-\text{V}(B) \leqslant \text{V}(A)$

But from Theorem 1, $\text{V}(A) + \text{V}(B) \leqslant 0$, so $\text{V}(A) \leqslant -\text{V}(B)$

Hence $\text{V}(A) = -\text{V}(B)$ and $\text{V}(A) + \text{V}(B) = 0$.

The results below follow directly from the stable solution theorem:

- **If a two-person zero-sum game has a stable solution:**
  - **it must occur when both players adopt their play-safe strategies**
  - **the value of the game to player $A$ is equal to the row maximin**
  - **the value of the game to player $B$ is equal to the negative of the column minimax**

You can use the stable solution theorem to show that a two-person zero-sum game has a stable solution.

## Example 4

A two-player zero-sum game has the following pay-off matrix for player $A$.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 7 | −4 | −2 |
| *A* plays 2 | −3 | 2 | 0 |
| *A* plays 3 | 5 | 5 | 6 |

**Online** Find stable solutions for zero-sum games using GeoGebra.

**a** Show that there is a stable solution.

**b** Find the value of the game to $A$.

**a** The first step is to find the row minimums and the column maximums.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 | Row minimum |
|---|---|---|---|---|
| *A* plays 1 | 7 | −4 | −2 | −4 |
| *A* plays 2 | −3 | 2 | 0 | −3 |
| *A* plays 3 | 5 | 5 | 6 | 5 |
| Column maximum | 7 | 5 | 6 | |

Row maximin = 5

Column minimax = 5

Row maximin = column minimax so by stable solution theorem the game has a stable solution.

**b** The value of the game to *A* is 5.

**Problem-solving**

The entry in row 3 and column 2 is a saddle point. If both players adopt their play-safe strategies, neither has any incentive to change, and player *A* will win 5.

## Example 5

A two-player zero-sum game has the following pay-off matrix:

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | 4 | −2 |
| *A* plays 2 | −5 | 3 |

**a** Determine the play-safe strategy for each player.

**b** Verify that there is no stable solution for this game.

**c** State the pay-off for each player if they both play safe.

**d** Determine the pay-off matrix for *B*.

**a**

| | *B* plays 1 | *B* plays 2 | Row minimum |
|---|---|---|---|
| *A* plays 1 | 4 | −2 | −2 |
| *A* plays 2 | −5 | 3 | −5 |
| Column maximum | 4 | 3 | |

The row maximin is −2 and the column minimax is 3.

*A*'s play-safe strategy is to play 1, and *B*'s play-safe strategy is to play 2.

**b** Since $-2 \neq 3$ the row maximin ≠ column minimax.
So by the stable solution theorem there is not a stable solution.

**Problem-solving**

Use the stable solution theorem. The fact that there is no stable solution means that the play-safe strategies for *A* and *B* cannot both be optimal. For example, if *B* plays 2, then *A* would increase his or her pay-off by playing 2.

**c** If *A* and *B* play safe, *A* plays 1 and *B* plays 2.

| | *B* plays 1 | *B* plays 2 | Row minimum |
|---|---|---|---|
| *A* plays 1 | 4 | −2 | −2 |
| *A* plays 2 | −5 | 3 | −5 |
| Column maximum | 4 | 3 | |

The pay-off for *A* is −2 and the pay-off for *B* is 2.

**d** The pay-off matrix for *B* is

| | *A* plays 1 | *A* plays 2 |
|---|---|---|
| *B* plays 1 | −4 | 5 |
| *B* plays 2 | 2 | −3 |

The values in the pay-off matrix are always given from the point of view of the 'row' player. To rewrite the pay-off matrix so it shows the winnings for player *B*, you need to transpose the matrix, then multiply each term by −1.

## Exercise 6A

**1** The pay-off matrix for a two-person game is given below.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | (2, 5) | (3, 1) | (2, 3) |
| ***A* plays 2** | (4, 1) | (3, 5) | (1, 2) |
| ***A* plays 3** | (3, 6) | (5, 4) | (7, 2) |
| ***A* plays 4** | (1, 4) | (5, 2) | (3, 4) |

**a** Explain why this is not a zero-sum game.

**b** State how much each player will win if *A* plays 2 and *B* plays 3.

**c** Determine the play-safe strategy for each player.

E **2** Alice and Bob play a zero-sum game. The game is represented by the following pay-off matrix for Alice.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | 5 | −3 | 8 |
| ***A* plays 2** | 4 | 5 | 7 |
| ***A* plays 3** | 6 | −5 | −4 |

**a** Explain what is meant by a play-safe strategy. **(2 marks)**

**b** State how much each player will win if *A* plays 2 and *B* plays 3. **(1 mark)**

**c** Find the row maximin and the column minimax. **(3 marks)**

**d** Hence determine the play-safe strategy for each player. **(1 mark)**

**e** State with reasons whether this game has a stable solution. **(1 mark)**

**3** Here is the pay-off matrix for a two-person zero-sum game.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 7 | −1 | 5 |
| *A* plays 2 | 3 | 6 | −4 |
| *A* plays 3 | −5 | −3 | 8 |
| *A* plays 4 | 4 | 1 | −3 |

**a** Find the play-safe strategy for each player.

**b** Determine the optimal strategy for player *A* if player *B* always plays safe.

**4** A two-person zero-sum game is represented by the following pay-off matrix for player *A*.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 3 | 2 | 3 |
| *A* plays 2 | −2 | 1 | 3 |
| *A* plays 3 | 4 | 2 | 1 |

**a** Determine the play-safe strategy for each player.

**b** Verify that there is a stable solution for this game and determine the saddle point.

**c** Write down the value of the game to player *A*.

**d** Write out the pay-off matrix from *B*'s point of view, and write down the value of the game to player *B*.

**5** Robert and Steve play a zero-sum game. This game is represented by the following pay-off matrix for Robert.

| | Steve plays 1 | Steve plays 2 | Steve plays 3 | Steve plays 4 |
|---|---|---|---|---|
| Robert plays 1 | −2 | −1 | −3 | 1 |
| Robert plays 2 | 2 | 3 | 1 | −2 |
| Robert plays 3 | 1 | 1 | −1 | 3 |

**a** Determine the play-safe strategy for each player.

**b** Verify that there is no stable solution for this game.

**c** Write out the pay-off matrix for Steve.

**Hint** The pay-off matrix for Steve will have 4 rows and 3 columns.

E/P **6** A two-person zero-sum game is represented by the following pay-off matrix for player *A*.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | −3 | −2 | 2 |
| *A* plays 2 | −1 | −1 | 3 |
| *A* plays 3 | 4 | −3 | 1 |
| *A* plays 4 | 3 | −1 | −1 |

**Problem-solving**

If the row maximin is the minimum value in more than one row, then either row represents a play-safe strategy for *A*. Similarly, if the column minimax is the maximum value in more than one column, either column represents a play-safe strategy for *B*.

**a** Determine the play-safe strategy, or strategies, for each player. **(4 marks)**

**b** Verify that there is a stable solution for this game and determine the saddle points. **(3 marks)**

**c** State the value of the game to player *A*. **(1 mark)**

(E) **7** Claire and David play a two-person zero-sum game, which is represented by the following pay-off matrix for Claire.

| | ***D* plays 1** | ***D* plays 2** | ***D* plays 3** | ***D* plays 4** |
|---|---|---|---|---|
| ***C* plays 1** | 7 | 2 | −3 | 5 |
| ***C* plays 2** | 4 | −1 | 1 | 3 |
| ***C* plays 3** | −2 | 5 | 2 | −1 |
| ***C* plays 4** | 3 | −3 | −4 | 2 |

**a** Explain what is meant by a zero-sum game. **(1 mark)**

**b** Determine the play-safe strategy for each player. **(3 marks)**

**c** Verify that there is no stable solution for this game. **(1 mark)**

**d** State the pay-off for Claire if both players play safe. **(1 mark)**

**e** State the pay-off for David if both players play safe. **(1 mark)**

**f** Determine the pay-off matrix for David. **(2 marks)**

(E/P) **8** Hilary and Denis play a two-person zero-sum game, which is represented by the following pay-off matrix for Hilary.

| | ***D* plays 1** | ***D* plays 2** | ***D* plays 3** | ***D* plays 4** | ***D* plays 5** |
|---|---|---|---|---|---|
| ***H* plays 1** | 2 | 1 | 0 | 0 | 2 |
| ***H* plays 2** | 4 | 0 | 0 | 0 | 2 |
| ***H* plays 3** | 1 | 4 | −1 | −1 | 3 |
| ***H* plays 4** | 1 | 1 | −1 | −2 | 0 |
| ***H* plays 5** | 0 | −2 | −3 | −3 | −1 |

**a** Define a saddle point for a two-person zero-sum game with a stable solution. **(1 marks**

**b** Determine the play-safe strategy, or strategies, for each player. **(3 marks)**

**c** Verify that there is a stable solution for this game and state the saddle points. **(3 marks)**

**d** State the value of the game for Hilary if both players play safe. **(1 mark)**

**e** State the value of the game for Denis if both players play safe. **(1 mark)**

**f** Determine the pay-off matrix for Denis. **(3 marks)**

(P) **9** Arjan and Beth are playing a game. Each player picks a number from 1 to 3. If they pick the same number, Arjan pays Beth that number in pounds. If they pick different numbers, Beth pays Arjan the difference between the two numbers in pounds.

**a** Explain why in the context of the question this is a zero-sum game.

**b** Write down Arjan's pay-off matrix.

**c** Verify that this game has no stable solution.

**d** Write down the pay-off matrix for Beth.

E/P **10** A two-person zero-sum game is represented by the following pay-off matrix, where $x$ is a real number:

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | 2 | 1 |
| *A* plays 2 | 5 | $x$ |
| *A* plays 3 | 0 | 3 |

Given that the game has a stable solution find

**a** the range of possible values of $x$ **(3 marks)**

**b** the value of the game, in terms of $x$. **(2 marks)**

**Problem-solving**
The position of the saddle point will depend on the value of $x$. For part **b**, you will need to consider two different situations.

**Challenge**
Prove that, if a two-person zero-sum game has multiple saddle points, then they must have the same value.

**Problem-solving** Let $x_{r,s}$ represent the entry in row $r$ and column $s$ of the pay-off matrix. Let $x_{a,b}$ and $x_{c,d}$ be saddle points, and consider $x_{a,d}$.

## 6.2 Reducing the pay-off matrix

A In some games, a particular choice by one player will **always** produce a worse outcome than another choice.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 3 | 0 | 1 |
| *A* plays 2 | 1 | −1 | 4 |
| *A* plays 3 | 5 | 2 | 2 |

Every entry in row 1 is lower than the corresponding entry in row 3. Row 3 would **always** be a better choice for $A$, regardless of the column that $B$ plays. You say that row 3 **dominates** row 1, and you can **reduce** the pay-off matrix by deleting row 1.

- **If every entry in row $p$ is greater than the corresponding entry in row $q$ then you can say that row $p$ dominates row $q$. In this case, row $q$ may be deleted as it would never be chosen.**
- **If every entry in column $r$ is less than the corresponding entry in column $s$ then you can say that column $r$ dominates column $s$. In this case column $s$ may be deleted as it would never be chosen.**

The column with the *smaller* values dominates since the negatives of those values represent the winnings of column player.

Deleting dominated rows or columns in this way is called reducing the pay-off matrix using **dominance arguments**.

## Example 6

A

*A* and *B* play a zero-sum game, represented by the pay-off matrix below.

Explain why the 3 × 2 game can be reduced to a 2 × 2 game.

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | 7 | 1 |
| *A* plays 2 | −3 | 6 |
| *A* plays 3 | 4 | 0 |

**Notation** This is a 3 × 2 game since player *A* has a choice of three rows, and player *B* has a choice of two columns.

If you look at *A*'s row 1 and row 3 choices you can see that *A* would never choose to play row 3, no matter which column *B* is playing.

Row 1 **dominates** row 3, since 7 > 4 and 1 > 0. Thus you can delete row 3 and reduce the game to a 2 × 2 game.

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | 7 | 1 |
| *A* plays 2 | −3 | 6 |

This means that, for player *A*, no matter what player *B* does, row 1 is **always** a better option than row 3.

**Watch out** You can only use dominance arguments to delete a 'named' row (or column) if **one** other 'named' row (column) is **always** better.

## Example 7

*A* and *B* play a zero-sum game, represented by the pay-off matrix below.

Explain why the 2 × 3 game can be reduced to a 2 × 2 game.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 7 | 1 | −2 |
| *A* plays 2 | −3 | 6 | 1 |

Column 3 is always a better choice than column 2 for player *B*. Remember that the numbers in the pay-off matrix show *A*'s winnings. *B* wishes to keep *A*'s winnings (and therefore his own losses) as small as possible. So *B* will never choose to play column 2, since, no matter what *A* does, column 3 gives a better result for *B*.

Column 3 dominates column 2 since −2 < 1 and 1 < 6. So you can delete column 2, reducing the game to a 2 × 2 game.

| | *B* plays 1 | *B* plays 3 |
|---|---|---|
| *A* plays 1 | 7 | −2 |
| *A* plays 2 | −3 | 1 |

**Watch out** Make sure you do not change the column headings. You have deleted column 2, so the new columns are headed '*B* plays 1' and '*B* plays 3'.

## Exercise 6B

1 Ellie and Freya play a zero-sum game, represented by the pay-off matrix for Ellie shown to the right. Use dominance to reduce the game to a 2 × 2 game. You must make your reasoning clear.

| | Freya plays 1 | Freya plays 2 |
|---|---|---|
| Ellie plays 1 | 1 | −5 |
| Ellie plays 2 | −1 | 6 |
| Ellie plays 3 | 3 | −3 |

A E 2

| | Harry plays 1 | Harry plays 2 | Harry plays 3 |
|---|---|---|---|
| **Doug plays 1** | −5 | 2 | −1 |
| **Doug plays 2** | 2 | −3 | −6 |

Doug and Harry play a zero-sum game, represented by the pay-off matrix for Doug shown above.

**a** Reduce the game so that Harry has a choice of only two actions. **(1 mark)**

**b** Write down the reduced pay-off matrix for Harry. **(2 marks)**

E/P 3

| | Nick plays 1 | Nick plays 2 | Nick plays 3 |
|---|---|---|---|
| **Chris plays 1** | 1 | 2 | 3 |
| **Chris plays 2** | −1 | −3 | 1 |
| **Chris plays 3** | 2 | −1 | 5 |

Chris and Nick play a zero-sum game, represented by the pay-off matrix for Chris shown above.

**a** Reduce the game so that each player only has two possible choices. **(2 marks)**

**b** Write down the reduced pay-off matrix for Nick. **(2 marks)**

E/P 4 Two businesses, Sakiya Inc. and Yin Industries, are planning to release competing products. Each company can make one of three marketing decisions. The outcomes of these decisions are modelled as a zero-sum game, represented by the following pay-off matrix for Sakiya (in £100 000 units):

| | Yin plays 1 | Yin plays 2 | Yin plays 3 |
|---|---|---|---|
| **Sakiya plays 1** | 4 | −6 | 2 |
| **Sakiya plays 2** | −1 | −12 | −3 |
| **Sakiya plays 3** | −5 | 7 | −8 |

**Problem-solving**

A dominated row or column cannot contain a saddle point. So if the reduced game has no stable solutions then neither does the original game.

**a** Use dominance to reduce the game so that Sakiya has a choice of only two actions. **(1 mark)**

**b** Determine the play-safe strategy for each business. **(3 marks)**

**c** Decide if there is a stable solution. Explain how you decided. **(2 marks)**

**d** Determine the profit or loss for Sakiya if both players play safe. **(2 marks)**

**e** Determine the profit or loss for Yin if both players play safe. **(1 mark)**

**f** Write down the reduced pay-off matrix for Yin. **(3 marks)**

E/P 5 A two-player zero-sum game is represented by the following pay-off matrix.

| | Brian plays 1 | Brian plays 2 | Brian plays 3 | Brian plays 4 |
|---|---|---|---|---|
| **Ali plays 1** | 0 | 7 | 2 | 12 |
| **Ali plays 2** | 9 | 8 | 8 | 10 |
| **Ali plays 3** | 10 | 4 | 3 | 0 |

**Notation** This is an example of **weak dominance**. If you remove a choice that is **never better** than another choice, then you can still find an optimal strategy. However, you may be removing an alternative optimal strategy.

Brian says that playing column 3 is always at least as good as playing column 2.

**a** Explain why Brian is correct. **(1 mark)**

A Brian reduces the pay-off matrix by removing column 2.

**b** **i** Write out the reduced pay-off matrix
**ii** Show that it has a saddle point and state the value of the game to Brian. **(4 marks)**

**c** By considering the original pay-off matrix, show that this game has more than one saddle point. **(2 marks)**

**Challenge**

A game $G$ is reduced by dominance arguments to a game $G'$.
Prove that $G$ has a stable solution if and only if $G'$ has a stable solution.

## 6.3 Optimal strategies for games with no stable solution

If a two-player zero-sum game has no stable solution, the players can improve their outcomes by occasionally deviating from their play-safe strategies.

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | 4 | −2 |
| *A* plays 2 | −5 | 3 |

The play-safe strategies are $A$ plays 1 and $B$ plays 2, but this game does not have a stable solution.

If $A$ and $B$ both play safe, then $A$ loses 2. But $A$ might be able to increase his or her winnings to 3 by occasionally playing row 2. Knowing this, $B$ might choose to occasionally play column 1, with the chance of increasing $A$'s losses to 5.

Each player can define a strategy such as this by assigning a **probability** to each available action. Every time the game is played, the player selects an action at random based on the assigned probabilities. Such a strategy is called a **mixed strategy**.

- **In a mixed strategy, each action is assigned a probability and is selected with the given probability.**

**Watch out** The probabilities of all the available actions must add up to 1.

For any given mixed strategy, you can calculate the **expected pay-off** for each of the opponents' possible actions.

**Example 8**

A two-player zero-sum game is represented by the following pay-off matrix.

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | 4 | −2 |
| *A* plays 2 | −5 | 3 |

Player $A$ is considering three different mixed strategies:

**Strategy *X*:** Play row 1 with probability 0.5 and row 2 with probability 0.5

**Strategy *Y*:** Play row 1 with probability 0.6 and row 2 with probability 0.4

**Strategy *Z*:** Play row 1 with probability 0.9 and row 2 with probability 0.1

Find the expected pay-offs for each mixed strategy.

**Strategy $X$:**
If $B$ plays 1, expected pay-off = $4 \times 0.5 - 5 \times 0.5 = -0.5$
If $B$ plays 2, expected pay-off = $-2 \times 0.5 + 3 \times 0.5 = 0.5$

**Strategy $Y$:**
If $B$ plays 1, expected pay-off = $4 \times 0.6 - 5 \times 0.4 = 0.4$
If $B$ plays 2, expected pay-off = $-2 \times 0.6 + 3 \times 0.4 = 0$

**Strategy $Z$:**
If $B$ plays 1, expected pay-off = $4 \times 0.9 - 5 \times 0.1 = 3.1$
If $B$ plays 2, expected pay-off = $-2 \times 0.9 + 3 \times 0.1 = -1.5$

| Strategy | If $B$ plays 1 | If $B$ plays 2 |
|---|---|---|
| $X$ | −0.5 | 0.5 |
| $Y$ | 0.4 | 0 |
| $Z$ | 3.1 | −1.5 |

The expected pay-off is the sum of the probabilities of each outcome multiplied by the value of that outcome. So, for example, with strategy $Y$, if $B$ plays 1 then $A$ will win 4 with probability 0.6 and −5 with probability 0.4.

With strategy $Y$, player $A$ can expect to either win 0.4 (if $B$ plays 1) or break even (if $B$ plays 2).

When choosing a play-safe strategy, players aim to maximise their minimum possible pay-off. Similarly, when choosing between different mixed strategies, players aim to maximise their minimum expected pay-off. In the above example, the minimum expected pay-offs are −0.5 (strategy $X$), 0 (strategy $Y$) and −1.5 (strategy $Z$). The strategy that maximises this value is strategy $Y$.

- **The optimal mixed strategy for player $A$ is the mixed strategy that maximises the minimum expected pay-off for all possible actions by player $B$.**

**Note** You can think of the optimal mixed strategy as the 'safest' mixed strategy.

All zero-sum games have an optimal mixed strategy for both players. The existence of optimal mixed strategies allows you to define the **value** of a game without a stable solution:

- **For a two-person zero-sum game with no stable solution, the value of the game to player $A$ is the expected pay-off for player $A$'s optimal mixed strategy.**

Because probabilities can vary continuously between 0 and 1, there are infinitely many possible mixed strategies. If you need to find an optimal mixed strategy for a player with **two possible actions** in a game, you can use the following **graphical approach**:

- **To find the optimal mixed strategy for a player with two choices in a $2 \times n$ or $n \times 2$ game:**
  - **Set the probability of one choice as $p$, so that the probability of the other choice is $1 - p$.**
  - **Find the expected winnings in terms of $p$.**
  - **Draw a graph showing the expected winnings as straight lines.**
  - **Choose the intersection point that maximises the minimum winnings.**

**Note** In your examination $n$ might be 2, 3 or 4. A-level students might need to use dominance arguments to reduce a pay-off matrix to one of these forms.

**Problem-solving**

If you always calculate the expected **winnings** for the player with two choices, then you can use a similar approach regardless of whether this is player $A$ or player $B$.

## Example 9

A two-person zero-sum game is represented by the following pay-off matrix:

| | ***B* plays 1** | ***B* plays 2** |
|---|---|---|
| ***A* plays 1** | 4 | −2 |
| ***A* plays 2** | −5 | 3 |

Find the optimal mixed strategy

**a** for player $A$ **b** for player $B$.

In each case state the value of the game to that player.

**a** For player $A$

Let $A$ play row 1 with a probability $p$, and therefore row 2 with a probability of $1 - p$.

$A$ knows that $B$ has two options:

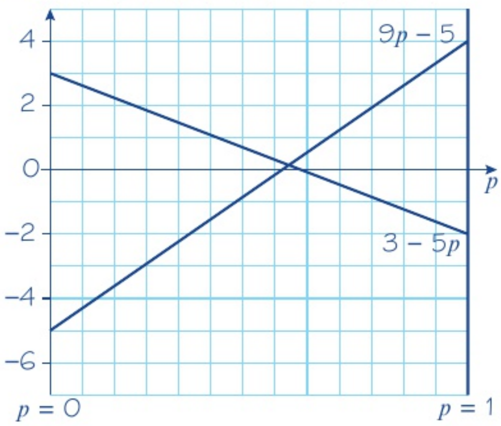
If $B$ plays column 1, $A$'s expected winnings are

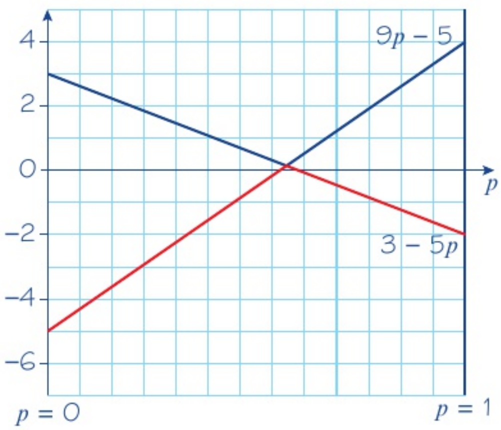
$4p - 5(1 - p) = 9p - 5$

If $B$ plays column 2, $A$'s expected winnings are

$-2p + 3(1 - p) = 3 - 5p$

Illustrate these options on a graph.




The point of intersection of the lines corresponds to the point where the minimum expected value of $A$'s earnings is maximised.

In game theory, not playing is not an option! So $A$ **must** play either row 1 or row 2, hence the probabilities **must** add up to 1. If $A$ plays row 1 with probability $p$, then the probability $A$ does not play row 1 (and so plays row 2) is $1 - p$.

Find the expected winnings **as a function of** $p$ for each of $B$'s possible choices.

Draw a graph with $p$ on the horizontal axis and expected winnings on the vertical axis. Each of your expressions for $A$'s expected winnings can be drawn as a straight line.

**Watch out** Make sure your lines stop at $p = 0$ and $p = 1$, because $p$ is a probability. The easiest way to do this is to draw vertical 'rugby posts' at $p = 0$ and $p = 1$.

Each line shows $A$'s expected winnings, as $p$ varies, depending on the option chosen by $B$.

In this diagram, the red line shows the minimum expected value of $A$'s winnings as $p$ varies.

At this point

$$9p - 5 = 3 - 5p$$
$$14p = 8$$
$$p = \tfrac{4}{7}$$

$A$ should play row 1 with a probability of $\frac{4}{7}$, and row 2 with a probability of $\frac{3}{7}$.

The value of the game for $A$ is $\frac{1}{7}$.

**b For player $B$**

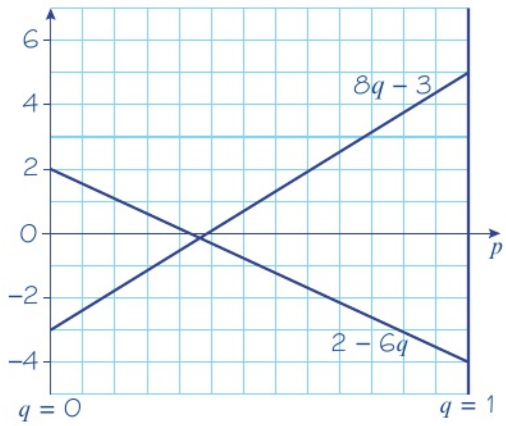
Similarly let $B$ play column 1 with a probabilityof $q$ and therefore column 2 with a probability of $1 - q$.

$B$ knows that $A$ has two options:

If $A$ plays row 1, $B$'s expected winnings are
$-[4q - 2(1 - q)] = 2 - 6q$

If $A$ plays row 2, $B$'s expected winnings are
$-[-5q + 3(1 - q)] = 8q - 3$

Use a graph to illustrate this information.




Finding the intersection gives $2 - 6q = 8q - 3$ and hence $q = \frac{5}{14}$.

Thus $B$ should play column 1 with a probability of $\frac{5}{14}$ and column 2 with a probability of $\frac{9}{14}$.

The value of the game for $B$ is $-\frac{1}{7}$.

**Problem-solving**

In Example 8 the 'safest' of the three proposed mixed strategies was the strategy with $p = 0.6$. You can see here that the optimal mixed strategy has a value of $p$ close to this value:
$p = \frac{4}{7} = 0.571\ldots$

You can calculate the value of the game to player $A$ by substituting the value of $p$ into either of the two equations you used to find its value.

Notice the minus signs outside the square brackets. The game is shown from player $A$'s point of view, so to find $B$'s winnings you need to change the signs.

**Problem-solving**

In a 2 × 2 game with no stable solution, the optimal mixed strategy for one player always occurs when the expected winnings for each of the other player actions are equal. However, in your examination you should always draw a graph when determining optimal mixed strategies.

Substitute $q$ into either of the two equations you used to calculate the intersection point to get the value of the game to player $B$.

In the above example, the value of the game to player $A$ was $\frac{1}{7}$ and the value of the game to player $B$ was $-\frac{1}{7}$. This relationship holds for all two-person zero-sum games.

- **In a two-person zero-sum game, if the value of the game to player $A$ is $v$, then the value of the game to player $B$ will be $-v$.**

**Note** This result was proved by John von Neumann in 1928 and is one of the most important results in game theory. One consequence of this result is that player $B$ could reveal his or her strategy to player $A$, and it would still not allow player $A$ to improve his or her expected outcome.

This means that if $A$'s optimal mixed strategy has a minimum expected pay-off of $v$, there exists an optimal mixed strategy for $B$ such that $B$'s expected losses are at most $v$.

## Example 10

Alf and Bert play a zero-sum game, represented by the pay-off matrix below.

| | Bert plays 1 | Bert plays 2 | Bert plays 3 |
|---|---|---|---|
| Alf plays 1 | −5 | 2 | −1 |
| Alf plays 2 | 2 | −3 | 1 |

**a** Verify that this game has no stable solution.

**b** Determine Alf's best strategy and the value of the game to him.

**a**

| | Bert plays 1 | Bert plays 2 | Bert plays 3 | Row minimum |
|---|---|---|---|---|
| Alf plays 1 | −5 | 2 | −1 | −5 |
| Alf plays 2 | 2 | −3 | 1 | −3 |
| Column maximum | 2 | 2 | 1 | |

The row maximin is −3 and the column minimax is 1.

$-3 \neq 1$ so by the stable solution theorem there is no stable solution.

**b** Alf has only two choices so it will be possible to determine his strategy.

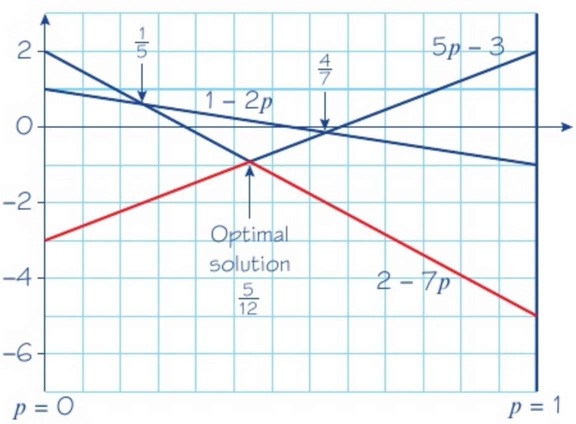
Let Alf play row 1 with probability $p$ and row 2 with probability $1 - p$.

If Bert plays column 1 Alf's expected winnings are $-5p + 2(1 - p) = 2 - 7p$

If Bert plays column 2 Alf's expected winnings are $2p - 3(1 - p) = 5p - 3$

If Bert plays column 3 Alf's expected winnings are $-p + (1 - p) = 1 - 2p$

If you illustrate these on a diagram you get:




You need to find the point of intersection that maximises the minimum winnings. This will be highest point of intersection **with no lines underneath it**.

**Watch out** You **must** draw a graph to identify the correct point of intersection. You should only need to solve **one** pair of equations to find $p$ after you have identified the correct point of intersection.

Now choose the intersection point that gives the highest minimum winnings.

This is where $2 - 7p = 5p - 3$, giving $p = \frac{5}{12}$.

So Alf should play row 1 with a probability of $\frac{5}{12}$, and row 2 with a probability of $\frac{7}{12}$. The value of the game to Alf is $2 - 7\left(\frac{5}{12}\right) = -\frac{11}{12}$.

- **To determine the optimal mixed strategy for the player with two choices in a $2 \times n$ game, where $n = 2, 3$ or $4$: if there is more than one intersection point the probability giving the highest minimum point gives the best strategy.**

## Example 11

$A$ and $B$ play a zero-sum game, given by the pay-off matrix below:

| | **B plays 1** | **B plays 2** |
|---|---|---|
| **A plays 1** | 4 | −1 |
| **A plays 2** | 3 | 4 |
| **A plays 3** | 1 | 7 |
| **A plays 4** | −2 | 8 |

Determine $B$'s best strategy, and the value of the game to $B$.

**Online** Explore the optimal solution to two-player zero-sum game with no stable solution using GeoGebra.

| | **B plays 1** | **B plays 2** | Row minimum |
|---|---|---|---|
| **A plays 1** | 4 | −1 | −1 |
| **A plays 2** | 3 | 4 | 3 |
| **A plays 3** | 1 | 7 | 1 |
| **A plays 4** | −2 | 8 | −2 |
| Column maximum | 4 | 8 | |

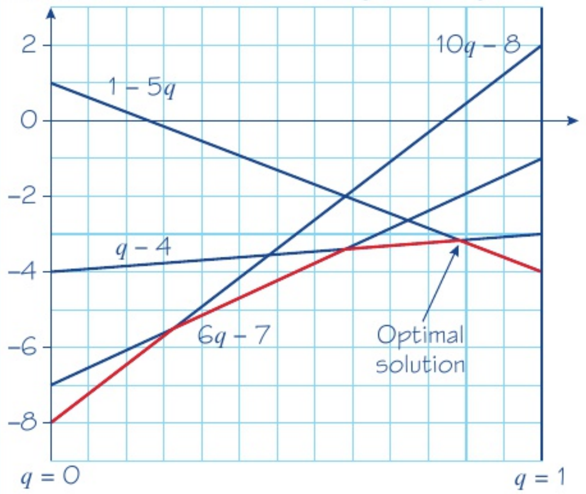
First check for a stable solution.
The row maximin is 3 and the column minimax is 4.
$3 \neq 4$ so there is no stable solution.

Player $B$ has only two choices so it will be possible to determine his strategy.
Let $B$ play column 1 with probability $q$ and column 2 with probability $1 - q$.

If $A$ plays row 1 $B$'s expected winnings are $-[4q - (1 - q)] = 1 - 5q$
If $A$ plays row 2 $B$'s expected winnings are $-[3q + 4(1 - q)] = q - 4$
If $A$ plays row 3 $B$'s expected winnings are $-[q + 7(1 - q)] = 6q - 7$
If $A$ plays row 4 $B$'s expected winnings are $-[-2q + 8(1 - q)] = 10q - 8$

The pay-off matrix is given from $A$'s point of view, so add minus signs to find $B$'s expected **winnings**.

If you illustrate these on a diagram you get:




**Watch out** The optimal solution is not necessarily the lowest point of intersection. Look from the 'bottom up' to find the highest point of intersection with no lines crossing below it.

Choose the intersection point that gives the highest minimum winnings.
This is where $1 - 5q = q - 4$, giving $q = \frac{5}{6}$.
So $B$ should play column 1 with a probability of $\frac{5}{6}$, and column 2 with a probability of $\frac{1}{6}$.
The value of the game to $B$ is $\frac{5}{6} - 4 = -\frac{19}{6}$.

## Exercise 6C

1 A two-person zero-sum game is represented by the following pay-off matrix for player *A*.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | –2 | 4 | 2 |
| *A* plays 2 | 0 | 5 | –1 |

Player *A* is considering five different mixed strategies:

**Strategy *V*:** Play row 1 with probability 0.5 and row 2 with probability 0.5

**Strategy *W*:** Play row 1 with probability 0.4 and row 2 with probability 0.6

**Strategy *X*:** Play row 1 with probability 0.3 and row 2 with probability 0.7

**Strategy *Y*:** Play row 1 with probability 0.2 and row 2 with probability 0.8

**Strategy *Z*:** Play row 1 with probability 0.1 and row 2 with probability 0.9

**a** Find the expected pay-offs for each mixed strategy.

Player *A* wants to choose the mixed strategy that maximises the worst possible pay-off.

**b** Which of the five mixed strategies given above should *A* choose? State the least possible expected pay-off for this strategy.

2 For each pay-off matrix,

**i** verify that there is no stable solution

**ii** determine the best strategy and the value of the game to player *A*

**iii** determine the best strategy and the value of the game to player *B*.

**a**

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | 2 | –4 |
| *A* plays 2 | –1 | 3 |

**b**

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | –3 | 5 |
| *A* plays 2 | 2 | –4 |

**c**

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | 5 | –1 |
| *A* plays 2 | –2 | 1 |

**d**

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | –1 | 3 |
| *A* plays 2 | 1 | –2 |

3 For each pay-off matrix,

**i** verify that there is no stable solution

**ii** determine the optimal mixed strategy and the value of the game to *A*.

**a**

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | –5 | 2 | 2 |
| *A* plays 2 | 1 | –3 | –4 |

**b**

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 2 | 6 | –2 |
| *A* plays 2 | –1 | –4 | 3 |

**c**

| | *B* plays 1 | *B* plays 2 | *B* plays 3 | *B* plays 4 |
|---|---|---|---|---|
| *A* plays 1 | −2 | 3 | 6 | 4 |
| *A* plays 2 | 5 | 1 | −4 | 6 |

**d**

| | *B* plays 1 | *B* plays 2 | *B* plays 3 | *B* plays 4 |
|---|---|---|---|---|
| *A* plays 1 | 5 | −2 | −4 | 6 |
| *A* plays 2 | −3 | 1 | 6 | 4 |

**4** For each pay-off matrix,

**i** verify that there is no stable solution

**ii** determine the optimal mixed strategy and the value of the game to *B*.

**a**

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | −1 | 1 |
| *A* plays 2 | 3 | −4 |
| *A* plays 3 | −2 | 2 |

**b**

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | −5 | 4 |
| *A* plays 2 | 3 | −3 |
| *A* plays 3 | 1 | −2 |

**c**

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | −3 | 2 |
| *A* plays 2 | −1 | −2 |
| *A* plays 3 | 2 | −4 |
| *A* plays 4 | 3 | −5 |

**d**

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | 2 | −3 |
| *A* plays 2 | −2 | 4 |
| *A* plays 3 | 1 | −1 |
| *A* plays 4 | −3 | 5 |

E **5** Two players play a zero-sum game. The pay-off matrix is given below.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 | *B* plays 4 |
|---|---|---|---|---|
| *A* plays 1 | −1 | 0 | −2 | 2 |
| *A* plays 2 | 1 | −2 | 3 | −3 |

**a** What is meant by a zero-sum game? **(1 mark)**

**b** Verify that there is no stable solution. **(2 marks)**

**c** Determine the optimal mixed strategy and the value of the game to *A*. **(7 marks)**

E/P **6** Two players play a zero-sum game. The pay-off matrix is given below.

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | −2 | 0 |
| *A* plays 2 | −1 | −3 |
| *A* plays 3 | −3 | 2 |
| *A* plays 4 | 1 | −4 |

**a** Verify that there is no stable solution. **(1 mark)**

**b** Determine the optimal mixed strategy and the value of the game to *B*. **(7 marks)**

**c** Find the value of the game to *A*. **(2 marks)**

E/P **7** Amy and Barun are playing a game. Each player has two cards, numbered 1 and 2. The players each choose one of their cards, and turn it face up.

- If the sum of the numbers on the two cards is **even** then Amy wins that amount in pounds.
- If the sum of the numbers on the two cards is **odd** then Barun wins that amount in pounds.

For example, if Amy plays 1 and Barun plays 2 then the total is 3. This is odd, so Barun wins £3.

**a** Draw the pay-off matrix for this game, from Amy's point of view. **(2 marks)**

**b** Verify that there is no stable solution and find Amy's optimal mixed strategy. **(4 marks)**

Amy says that the game is biased in Barun's favour, and that to make the game fair, he should pay her 10p every time they play.

**Problem-solving**

For part **c**, find the value of the game to Amy, and write a conclusion.

**c** Comment on Amy's proposal. **(3 marks)**

A E **8** A two-person zero-sum game is represented by the following pay-off matrix for player $A$.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | 6 | −2 | 2 |
| ***A* plays 2** | 1 | 4 | 5 |
| ***A* plays 3** | −3 | −4 | 1 |

**Problem-solving**

In an optimal strategy, player $A$ should never play a dominated row. This means that an optimal mixed strategy for the reduced pay-off matrix will also be an optimal mixed strategy for the original game. The dominated row can be assigned probability 0.

**a** Verify that the game has no stable solution. **(2 marks)**

**b** Use dominance to simplify the pay-off matrix. **(3 marks)**

**c** Determine the optimal mixed strategy and find the value of the game to $A$. **(5 marks)**

E **9** A two-person zero-sum game is represented by the following pay-off matrix for player $A$.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | −4 | 2 | 3 |
| ***A* plays 2** | −1 | 3 | 0 |
| ***A* plays 3** | 1 | 6 | −4 |

**a** Define the term 'saddle point'. **(1 mark)**

**b** State the number of points that player $B$ gets if $B$ plays 3 and $A$ plays 3. **(1 mark)**

**c** Reduce the game so that player $B$ only has two choices to play. **(3 marks)**

**d** Find the best strategy for player $B$ to play. **(5 marks)**

**e** What is the value of the game to him if he plays this strategy? **(1 mark)**

A E/P **10** A two-person zero-sum game is represented by the following pay-off matrix.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | 5 | 2 | 3 |
| ***A* plays 2** | −3 | 1 | −1 |
| ***A* plays 3** | −1 | 4 | −2 |

Find the best strategy for player $A$, and state the value of the game to $A$.
Show enough working to fully justify your answer. **(10 marks)**

E/P **11** Alice and Bob play a zero-sum game. The pay-off matrix for Alice is shown below.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | −2 | 3 | 4 |
| ***A* plays 2** | 1 | −2 | −1 |
| ***A* plays 3** | −4 | 5 | 6 |

Bob claims that he can find a strategy that will give him expected winnings of $\frac{1}{2}$.
Is Bob correct? Show all of your working. **(10 marks)**

**Challenge**

A two-person zero-sum game is represented by the following pay-off matrix, where $x$ is a real number:

| | ***B* plays 1** | ***B* plays 2** |
|---|---|---|
| ***A* plays 1** | 1 | $x$ |
| ***A* plays 2** | 3 | 0 |

Find, as a function of $x$, the value of the game to player $A$.

**Problem-solving**

You will need to consider separately the values of $x$ for which the game has a stable solution.

## 6.4 Converting games to linear programming problems

A It is possible to formulate a game as a linear programming problem. This approach is used when the game cannot be reduced to a game in which one player has just two options.

By convention, the variables in a linear programming problem must all take positive values. You need to transform a game by adding a constant to each value in the pay-off matrix to make every entry positive. This transformation will not affect the optimal strategy, and you can find the value of the original game by subtracting the same constant at the end.

**Links** Once you have formulated a game as a linear programming problem, it can be solved using the simplex algorithm. ← D1, Chapter 8

**Watch out** You would normally check for stable solutions, and to see whether the pay-off matrix can be reduced using dominance arguments, before attempting to formulate a game as a linear programming problem.

- **To formulate a game as a linear programming problem:**
  - **Transform the game by adding $n$ to each value so that all of the values are non-negative.**
  - **Define the decision variables.**
  - **Write $V$ (the value of the new game, with $n$ added) and set the objective to maximise $V$.**
  - **Write down the constraints.**

**Watch out** You should only write down the simplex tableau or attempt to solve the problem if you are told to do so in the question.

## Example 12

Alf and Bert play a zero-sum game, represented by the pay-off matrix below.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | −2 | 4 | 2 |
| *A* plays 2 | 0 | −3 | −2 |
| *A* plays 3 | −6 | 1 | 3 |

**a** Formulate the game above as a linear programming problem for player $A$, writing the constraints as equations and defining your variables clearly.

**b** Write down an initial simplex tableau for this problem.

**c** Outline how you would change the method to formulate the game as a linear programming problem for player $B$. (You do not need to formulate the problem.)

**a** Formulate the game for player $A$

| | *B* plays 1 | *B* plays 2 | *B* plays 3 | Row minimum |
|---|---|---|---|---|
| *A* plays 1 | −2 | 4 | 2 | −2 |
| *A* plays 2 | 0 | −3 | −2 | −3 |
| *A* plays 3 | −6 | 1 | 3 | −6 |
| Column maximum | 0 | 4 | 3 | |

The row maximin is −2 and the column minimax is 0.
Since $-2 \neq 0$, there is no stable solution.
You cannot use row or column dominance to remove any rows or columns.

Adding 7 to each element gives

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 5 | 11 | 9 |
| *A* plays 2 | 7 | 4 | 5 |
| *A* plays 3 | 1 | 8 | 10 |

First transform the game by adding $n$ to each element, to make the values non-negative. This ensures that the feasible region will be completely in the positive 'quadrant'.

In this case 7 has been added to every element.

**Note** You could also add 6 to each value. However, if you make all the values **strictly positive** then $p_1$, $p_2$ and $p_3$ all appear in every constraint. This makes it easier to check your constraints.

Let $p_1$ be the probability of $A$ playing row 1
Let $p_2$ be the probability of $A$ playing row 2
Let $p_3$ be the probability of $A$ playing row 3
where $p_1, p_2, p_3 \geqslant 0$.

Then define your decision variables. These are the probabilities assigned to each action in player $A$'s mixed strategy.

Let $v$ = value of the original game to player $A$.

Then $V = v + 7$ = value of the new game to player $A$.

**Maximise** $P = V$ so $P - V = 0$

Next write down the objective function.

subject to

If $B$ plays column 1, $5p_1 + 7p_2 + p_3 \geqslant V$

If $B$ plays column 2, $11p_1 + 4p_2 + 8p_3 \geqslant V$

If $B$ plays column 3, $9p_1 + 5p_2 + 10p_3 \geqslant V$

$p_1 + p_2 + p_3 = 1$

where $p_1, p_2, p_3 \geqslant 0$

$V$ is the minimum that $A$ can expect to win, so his expected winnings are $\geqslant V$.

Finally write down the constraints.

So

$$V - 5p_1 - 7p_2 - p_3 \leqslant 0$$
$$V - 11p_1 - 4p_2 - 8p_3 \leqslant 0$$
$$V - 9p_1 - 5p_2 - 10p_3 \leqslant 0$$
$$p_1 + p_2 + p_3 \leqslant 1$$

Notice that we have replaced $p_1 + p_2 + p_3 = 1$ by the weaker constraint $p_1 + p_2 + p_3 \leqslant 1$ so that the simplex algorithm can be applied.

**b** The linear program is

Maximise $P = V$

subject to

$$V - 5p_1 - 7p_2 - p_3 + r = 0$$
$$V - 11p_1 - 4p_2 - 8p_3 + s = 0$$
$$V - 9p_1 - 5p_2 - 10p_3 + t = 0$$
$$p_1 + p_2 + p_3 + u = 1$$
$$p_1, p_2, p_3, r, s, t, u \geqslant 0$$

The initial simplex tableau is:

| Basic variable | $V$ | $p_1$ | $p_2$ | $p_3$ | $r$ | $s$ | $t$ | $u$ | Value |
|---|---|---|---|---|---|---|---|---|---|
| $r$ | 1 | −5 | −7 | −1 | 1 | 0 | 0 | 0 | 0 |
| $s$ | 1 | −11 | −4 | −8 | 0 | 1 | 0 | 0 | 0 |
| $t$ | 1 | −9 | −5 | −10 | 0 | 0 | 1 | 0 | 0 |
| $u$ | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| $P$ | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**c** To formulate the game for player $B$:

1 Rewrite the matrix from player $B$'s point of view.
- Transpose the matrix so all the rows become columns, and all the columns become rows.
- Change the signs so positives become negatives, and negatives positives.

In this example the game becomes

| | $A$ plays 1 | $A$ plays 2 | $A$ plays 3 |
|---|---|---|---|
| $B$ plays 1 | 2 | 0 | 6 |
| $B$ plays 2 | −4 | 3 | −1 |
| $B$ plays 3 | −2 | 2 | −3 |

2 Continue with the method as usual:
- make all the entries positive
- define the variables
- write down the objective and constraints.

## Example 13

A

Jenny and Merry play a zero-sum game, represented by the pay-off matrix for Jenny shown below.

| | Merry plays 1 | Merry plays 2 | Merry plays 3 |
|---|---|---|---|
| **Jenny plays 1** | −3 | 0 | 2 |
| **Jenny plays 2** | −2 | 2 | 1 |
| **Jenny plays 3** | 0 | −3 | −2 |

**a** Convert the game into a linear programming problem for Jenny, defining your decision variables. You should write the constraints as equations.

**b** Write down an initial simplex tableau, making your variables clear.

**c** Using the simplex algorithm, the solution obtained is $p_1 = 0$, $p_2 = \frac{3}{7}$, $p_3 = \frac{4}{7}$
Find the value of the game to Jenny.

a

| | Merry plays 1 | Merry plays 2 | Merry plays 3 | Row minimum |
|---|---|---|---|---|
| Jenny plays 1 | −3 | 0 | 2 | −3 |
| Jenny plays 2 | −2 | 2 | 1 | −2 |
| Jenny plays 3 | 0 | −3 | −2 | −3 |
| Column maximum | 0 | 2 | 2 | |

The row maximin is −2 and the column minimax is 0.

$-2 \neq 0$ so there is no stable solution.

You cannot use row or column dominance to remove any rows or columns.

In this case, add 4 to every element.

First transform the game by adding $n$ to each element, to make the values positive ($>0$).

So the game becomes:

| | $M$ plays 1 | $M$ plays 2 | $M$ plays 3 |
|---|---|---|---|
| $J$ plays 1 | 1 | 4 | 6 |
| $J$ plays 2 | 2 | 6 | 5 |
| $J$ plays 3 | 4 | 1 | 2 |

**Watch out** At the end you **must** remember to subtract this constant from the number that the tableau gives to find the value of the game.

Let Jenny play row 1 with probability $p_1$, row 2 with probability $p_2$ and row 3 with probability $p_3$.

$p_1 + p_2 + p_3 = 1$

If Merry plays column 1, Jenny's expected winnings are $p_1 + 2p_2 + 4p_3$

If Merry plays column 2, Jenny's expected winnings are $4p_1 + 6p_2 + p_3$

If Merry plays column 3, Jenny's expected winnings are $6p_1 + 5p_2 + 2p_3$

Now if you let $V$ be the value of the transformed game, Jenny will be looking for probabilities such that

$$V \leqslant p_1 + 2p_2 + 4p_3$$
$$V \leqslant 4p_1 + 6p_2 + p_3$$
$$V \leqslant 6p_1 + 5p_2 + 2p_3$$

Rearrange these inequalities and introduce slack variables $r$, $s$ and $t$.

Maximise $P = V$

subject to

$$V - p_1 - 2p_2 - 4p_3 + r = 0$$
$$V - 4p_1 - 6p_2 - p_3 + s = 0$$
$$V - 6p_1 - 5p_2 - 2p_3 + t = 0$$
$$p_1 + p_2 + p_3 + u = 1$$
$$p_1, p_2, p_3, r, s, t, u \geqslant 0$$

$u$ is really just a check variable.
At the end the final tableau must give $u = 0$.
The probabilities must total 1. In maximising $V$ you will get $p_1 + p_2 + p_3 = 1$.

**b** The initial tableau looks like this.

| Basic variable | $V$ | $p_1$ | $p_2$ | $p_3$ | $r$ | $s$ | $t$ | $u$ | Value |
|---|---|---|---|---|---|---|---|---|---|
| $r$ | 1 | −1 | −2 | −4 | 1 | 0 | 0 | 0 | 0 |
| $s$ | 1 | −4 | −6 | −1 | 0 | 1 | 0 | 0 | 0 |
| $t$ | 1 | −6 | −5 | −2 | 0 | 0 | 1 | 0 | 0 |
| $u$ | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| $P$ | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**c** Substituting the probability values in the inequalities for $V$ gives

$$V \leqslant 2 \times \frac{3}{7} + 4 \times \frac{4}{7} = \frac{22}{7}$$
$$V \leqslant 6 \times \frac{3}{7} + 1 \times \frac{4}{7} = \frac{22}{7}$$
$$V \leqslant 5 \times \frac{3}{7} + 2 \times \frac{4}{7} = \frac{23}{7}$$

The minimum of these is $\frac{22}{7}$. This is the value of the transformed game.

The value of the game to Jenny is $\frac{22}{7} - 4 = -\frac{6}{7}$.

Subtract 4 because 4 was added to each term in part **b**.

## Exercise 6D

**1** Formulate the games below as linear programming problems for player $A$, writing the constraints as equalities and clearly defining your variables.

**a**

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | −5 | 4 | 1 |
| ***A* plays 2** | 3 | −3 | 2 |
| ***A* plays 3** | 1 | −2 | −1 |

**b**

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | −3 | 2 | −1 |
| ***A* plays 2** | −1 | −2 | 1 |
| ***A* plays 3** | 2 | −4 | −2 |

A E **2**

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 2 | −3 | −1 |
| *A* plays 2 | −2 | 4 | 1 |
| *A* plays 3 | 1 | −1 | 0 |

A two-person zero-sum game is represented by this pay-off matrix.
Formulate the game as a linear programming problem for player *A*.
Define your variables and write the constraints as inequalities. **(7 marks)**

**3** Formulate the games below as linear programming problems for player *B*, writing the constraints as equalities and clearly defining your variables.

**a**

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | −5 | 4 | 1 |
| *A* plays 2 | 3 | −3 | 2 |
| *A* plays 3 | 1 | −2 | −1 |

**b**

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | −3 | 2 | −1 |
| *A* plays 2 | −1 | −2 | 1 |
| *A* plays 3 | 2 | −4 | −2 |

**c**

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 2 | −3 | −1 |
| *A* plays 2 | −2 | 4 | 1 |
| *A* plays 3 | 1 | −1 | 0 |

**4** A two-player zero-sum game is defined by the following pay-off matrix.

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | −1 | 1 |
| *A* plays 2 | 3 | −4 |
| *A* plays 3 | −2 | 2 |

**a** Formulate this game as a linear programming problem for player *A*, writing the constraints as equalities and clearly defining your variables.

**b** Write down an initial simplex tableau to solve the problem.

**c** Given that the best strategy for *A* is to play 1 with probability $\frac{7}{9}$ and to play 2 with probability $\frac{2}{9}$, find the value of the game to player *A*.

E/P **5** **a** Formulate the game below as a linear programming problem for player *B*, writing the constraints as equalities and clearly defining your variables.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | −5 | 2 | 3 |
| *A* plays 2 | 1 | −3 | −4 |

**(7 marks)**

A **b** Write down an initial simplex tableau to solve the zero-sum game above for player *B*. **(3 marks)**

**c** Given that the best strategy for *B* is to play 1 with probability $\frac{7}{13}$ and to play 3 with probability $\frac{6}{13}$, find the value of the game to player *B*. **(2 marks)**

E/P **6** A two-player zero-sum game has the following pay-off matrix for player *A*.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | 5 | 3 | −1 |
| ***A* plays 2** | 4 | 5 | 2 |
| ***A* plays 3** | −4 | 2 | −3 |
| ***A* plays 4** | 7 | −2 | 4 |

**a** Verify that the game does not have a stable solution. **(3 marks)**

**b** Use dominance to reduce the number of choices available to player *A* to 3. **(2 marks)**

**c** Formulate the game as a linear programming problem for player *A*. Define the variables and write the constraints as inequalities. **(7 marks)**

## Mixed exercise 6

E/P **1** A two-person zero-sum game is represented by the following pay-off matrix for player *A*. Find the best strategy for each player and the value of the game.

| | ***B* plays 1** | ***B* plays 2** |
|---|---|---|
| ***A* plays 1** | 4 | −2 |
| ***A* plays 2** | −5 | 6 |

**(7 marks)**

E/P **2** Marriette and Nigel play a zero-sum game. The following pay-off matrix for Marriette shows the game.

| | **Nigel plays 1** | **Nigel plays 2** | **Nigel plays 3** | **Nigel plays 4** |
|---|---|---|---|---|
| **Marriette plays 1** | −1 | −4 | −2 | 2 |
| **Marriette plays 2** | 1 | −3 | 0 | 3 |
| **Marriette plays 3** | 2 | −5 | −2 | 4 |

**a** Nigel says that a zero-sum game must have a value of zero. Is he correct? Give a reason for your answer. **(1 mark)**

**b** Show that there is a stable solution to the game. **(2 marks)**

**c** Find the play-safe strategy for each player and the value of the game to Marriette. **(3 marks)**

E **3** A two-person zero-sum game is represented by the following pay-off matrix for player *A*.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | −3 | −2 | 2 |
| ***A* plays 2** | 1 | −4 | 3 |
| ***A* plays 3** | 4 | −3 | 1 |
| ***A* plays 4** | 2 | −1 | 3 |

**a** What is meant by a 'play-safe strategy'? **(1 mark)**

**b** Determine the play-safe strategy for both players. **(2 marks)**

c Show that there is a stable solution to this game. **(2 marks)**

d Determine the saddle point. **(1 mark)**

e Determine the value of the game to player $A$. **(1 mark)**

E **4** Tadashi and Molly play a zero-sum game, represented by the following pay-off matrix for Tadashi.

| | **Molly plays 1** | **Molly plays 2** |
|---|---|---|
| **Tadashi plays 1** | 5 | −2 |
| **Tadashi plays 2** | −3 | 4 |

a Verify that there is no stable solution to the game. **(3 marks)**

b Find the best strategy for Tadashi and his expected winnings. **(4 marks)**

E **5** Two players play a zero-sum game, represented by the pay-off matrix given below.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | 7 | −6 | 1 |
| ***A* plays 2** | −1 | 7 | 0 |

a Verify that there is no stable solution to the game. **(3 marks)**

b Use a graphical method to determine $A$'s optimal mixed strategy. **(3 marks)**

c Determine the value of the game to player $A$. **(2 marks)**

E/P **6** Olivia and Jacob play a zero-sum game that will require a mixed-strategy solution. The pay-off matrix is given below.

| | **Jacob plays 1** | **Jacob plays 2** | **Jacob plays 3** | **Jacob plays 4** |
|---|---|---|---|---|
| **Olivia plays 1** | 2 | 4 | −2 | −2 |
| **Olivia plays 2** | −2 | −1 | 2 | 1 |

a Explain the difference between a pure strategy and a mixed strategy. **(2 marks)**

b Using a graphical method, find the best strategy for Olivia to play. **(7 marks)**

c Find the value of the game to Olivia. **(2 marks)**

A E **7** Ben and Greg play a zero-sum game, represented by the following pay-off matrix for Ben.

| | **Greg plays 1** | **Greg plays 2** | **Greg plays 3** |
|---|---|---|---|
| **Ben plays 1** | −5 | 4 | 3 |
| **Ben plays 2** | 1 | −1 | −4 |

a Explain why this matrix can be reduced to

| | **Greg plays 1** | **Greg plays 3** |
|---|---|---|
| **Ben plays 1** | −5 | 3 |
| **Ben plays 2** | 1 | −4 |

**(2 marks)**

b Hence find the best strategy for each player and the value of the game. **(3 marks)**

A E **8** Cait and Georgi play a zero-sum game, represented by the following pay-off matrix for Cait.

| | **Georgi plays 1** | **Georgi plays 2** | **Georgi plays 3** |
|---|---|---|---|
| **Cait plays 1** | −5 | 2 | 3 |
| **Cait plays 2** | 1 | −3 | −4 |
| **Cait plays 3** | −7 | 0 | 1 |

**a** Identify the play-safe strategies for each player. **(3 marks)**

**b** Verify that there is no stable solution to this game. **(2 marks)**

**c** Use dominance to reduce the game to a 2 × 3 game, explaining your reasoning. **(2 marks)**

**d** Find Cait's best strategy and the value of the game to her. **(3 marks)**

**e** Write down the value of the game to Georgi. **(1 mark)**

**9** A two-person zero-sum game is represented by the following pay-off matrix for player $A$.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | 2 | −1 | −3 |
| ***A* plays 2** | −2 | 1 | 4 |
| ***A* plays 3** | −3 | 1 | −3 |
| ***A* plays 4** | −1 | 2 | −2 |

**a** Verify that there is no stable solution to this game

**b** Explain the circumstances under which a row, $x$, dominates a row, $y$.

**c** Reduce the game to a 3 × 3 game, explaining your reasoning.

**d** Formulate the 3 × 3 game as a linear programming problem for player $A$. Write the constraints as inequalities and define your variables.

E **10** A two-person zero-sum game is represented by the following pay-off matrix for player $A$.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | 5 | −3 | 1 |
| ***A* plays 2** | −1 | −4 | 4 |
| ***A* plays 3** | 3 | 2 | −1 |

**a** Identify the play-safe strategies for each player. **(2 marks)**

**b** Verify that there is no stable solution to this game. **(2 marks)**

**c** Use dominance to reduce the game to a 3 × 2 game, explaining your reasoning. **(3 marks)**

**d** Write down the pay-off matrix for player $B$. **(1 mark)**

**e** Find $B$'s best strategy and the value of the game. **(4 marks)**

E **11** A two-person zero-sum game is represented by the following pay-off matrix for player $A$.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** |
|---|---|---|---|
| ***A* plays 1** | 2 | 7 | −1 |
| ***A* plays 2** | 5 | 0 | 8 |
| ***A* plays 3** | −2 | 3 | 5 |

A

**a** Identify the play-safe strategies for each player. **(2 marks)**

**b** Verify that there is no stable solution to this game. **(2 marks)**

**c** Write down the pay-off matrix for player $B$. **(1 mark)**

**d** Formulate the game for player $B$ as a linear programming problem. Define your variables and write your constraints as equations. **(5 marks)**

**e** Write down an initial simplex tableau that you could use to solve the game for player $B$. **(2 marks)**

## Challenge

A two-person zero-sum game is represented by the following pay-off matrix, where $a$, $b$, $c$ and $d$ are real numbers.

| | $B$ plays 1 | $B$ plays 2 |
|---|---|---|
| $A$ plays 1 | $a$ | $b$ |
| $A$ plays 2 | $c$ | $d$ |

**a** Given that the game does not have a stable solution, find optimal mixed strategies for each player, giving your probabilities in terms of $a$, $b$, $c$ and $d$.

**b** Find the value of the game to each player, giving your answer in terms of $a$, $b$, $c$ and $d$.

**c** Hence verify that (value to player $A$) = –(value to player $B$).

## Summary of key points

1 A **two-person game** is one in which only two parties can play.

2 When **playing safe** each player looks for the worst that could happen if they make each choice in turn. The player picks the choice that results in the least worst option.

3 A strategy in which a player always makes the same choice is called a **pure strategy**.

4 In a **zero-sum game** the two entries in each cell in the pay-off matrix add up to zero.

5 A pay-off matrix for a zero-sum game uses only one value in each position. It is always written from the row player's ($A$'s) point of view unless you are told otherwise.

6 The **play-safe strategies** are:

- For player $A$, always play the row containing the row **maximin**.
- For player $B$, always play the column containing the column **minimax**.

7 A two-person zero-sum game has a **stable solution** if there exist pure strategies for both players in which neither player has an incentive to change their strategy.

8 A **saddle point** in a pay-off matrix is a value which is the smallest in its row and the largest in its column.

**9** A two-person zero-sum game has a stable solution if and only if its pay-off matrix contains a saddle point.

**10** In a two-person zero-sum game there will be a stable solution if and only if:

the row maximin = the column minimax

**11** If a two-person zero-sum game has a stable solution:

- it must occur when both players adopt their play-safe strategies
- the value of the game to player $A$ is equal to the row maximin
- the value of the game to player $B$ is equal to the negative of the column minimax.

A

**12**
- If every entry in row $p$ is greater than the corresponding entry in row $q$ then you can say that row $p$ dominates row $q$. In this case, row $q$ may be deleted as it would never be chosen.
- If every entry in column $r$ is less than the corresponding entry in column $s$ then you can say that column $r$ dominates column $s$. In this case column $s$ may be deleted as it would never be chosen.
- Deleting dominated rows or columns in this way is called **reducing the pay-off matrix** using dominance arguments.

**13** In a mixed strategy, each action is assigned a probability and is selected with the given probability.

**14** The optimal mixed strategy for player $A$ is the mixed strategy that maximises the minimum expected pay-off for all possible actions by player $B$.

**15** For a two-person zero-sum game with no stable solution, the value of the game to player $A$ is the expected pay-off for player $A$'s optimal mixed strategy.

**16** To find the optimal mixed strategy for a player with two choices in a $2 \times n$ or $n \times 2$ game:

- Set the probability of one choice as $p$, so that the probability of the other choice is $1 - p$.
- Find the expected winnings in terms of $p$.
- Draw a graph showing the expected winnings as straight lines.
- Choose the intersection point which maximises the minimum winnings.

**17** In a two-person zero-sum game, if the value of the game to player $A$ is $v$, then the value of the game to player $B$ will be $-v$.

**18** To determine the optimal mixed strategy for the player with two choices in a $2 \times n$ game, where $n = 2$, 3 or 4: if there is more than one intersection point, the probability giving the highest minimum point gives the best strategy.

A

**19** To formulate a game as a linear programming problem:

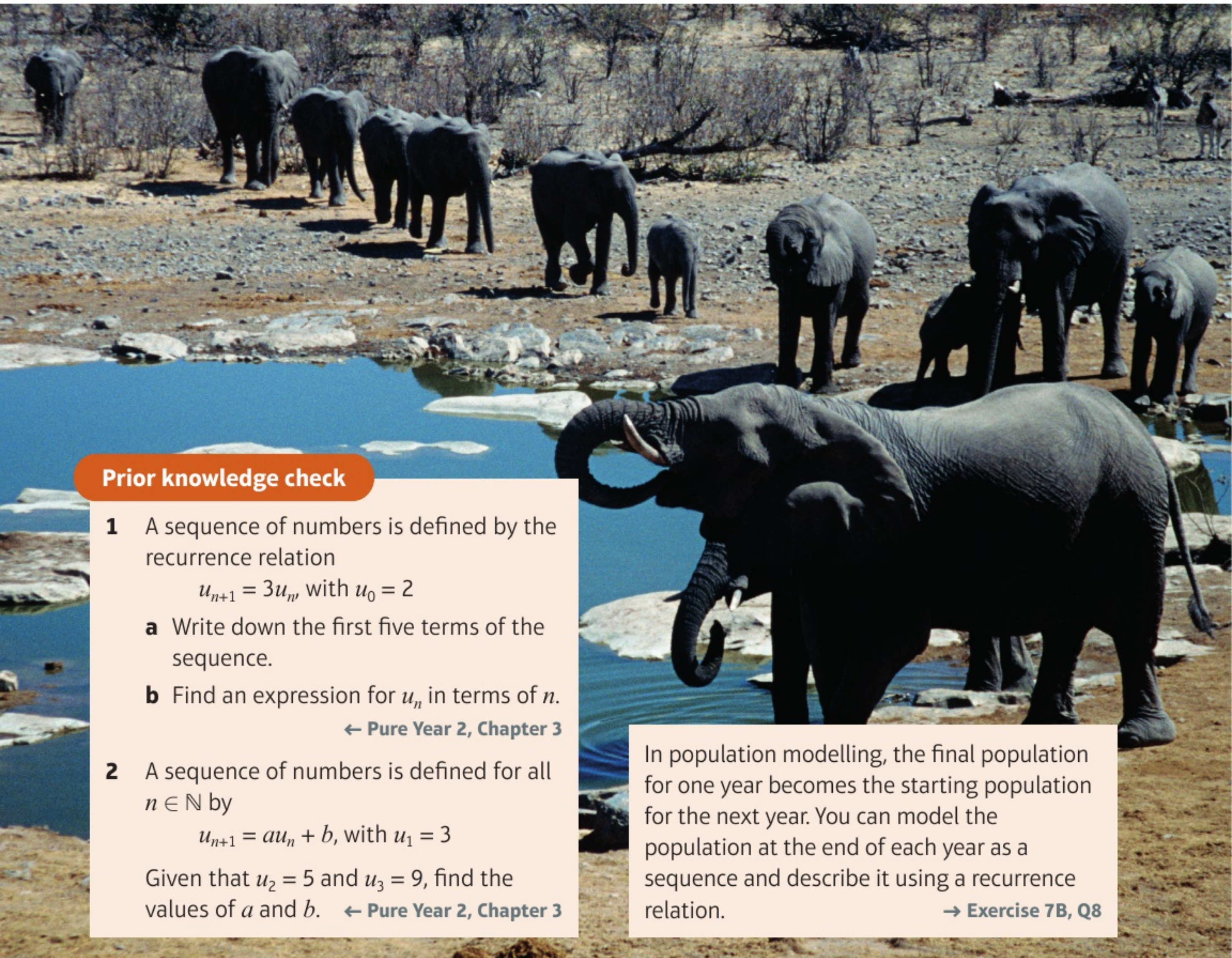
- Transform the game by adding $n$ to each value so that all of the values are non-negative.
- Define the decision variables.
- Write $V$ (the value of the new game, with $n$ added) and set the objective to maximise $V$.
- Write down the constraints.

# 7 Recurrence relations

**Objectives**

After completing this chapter you should be able to:

- Use recurrence relations to describe sequences and model situations → pages 215–219
- Find solutions to first-order recurrence relations → pages 219–227
- Find solutions to second-order recurrence relations → pages 228–235



**Prior knowledge check**

1 A sequence of numbers is defined by the recurrence relation

$u_{n+1} = 3u_n$, with $u_0 = 2$

**a** Write down the first five terms of the sequence.

**b** Find an expression for $u_n$ in terms of $n$.

← Pure Year 2, Chapter 3

2 A sequence of numbers is defined for all $n \in \mathbb{N}$ by

$u_{n+1} = au_n + b$, with $u_1 = 3$

Given that $u_2 = 5$ and $u_3 = 9$, find the values of $a$ and $b$. ← Pure Year 2, Chapter 3

In population modelling, the final population for one year becomes the starting population for the next year. You can model the population at the end of each year as a sequence and describe it using a recurrence relation. → Exercise 7B, Q8

## 7.1 Forming recurrence relations

You can model many real-life situations using recurrence relations.

For example, suppose that you have £500 in a savings account that pays 0.5% interest every month. Each month, you add another £100 to the savings account.

**Links** A recurrence relation describes each term of a sequence in terms of the previous term or terms. ← Pure Year 2, Section 3.7

You can use this information to formulate a recurrence relation that describes the amount in the account at the end of each month.

Let $u_m$ be the amount in pounds in the account after $m$ months. The next month, $m + 1$, you will have the original amount, $u_m$, plus the interest, $0.005u_m$, plus the additional £100 you add every month. This generates the recurrence relation

$$u_{m+1} = u_m + 0.005u_m + 100, \text{ with } u_0 = 500$$

You need to give the initial amount in the account to fully define the sequence. This is sometimes called an **initial condition** for the recurrence relation.

**Notation** This is an example of a **first-order recurrence relation,** as $u_{m+1}$ is given in terms of **one previous term**, $u_m$. → Section 7.2

### Example 1

Harry owes £500 on a credit card that charges 1.5% interest each month. He decides to make no new charges and pays off £50 each month. Formulate a recurrence relation that describes the balance remaining on the credit card after $n$ months.

Let $u_n$ be the amount in pounds owed after $n$ months.

*Define the terms and give any relevant units.*

During a month, the interest is $0.015u_n$ and you pay off £50.

*The interest is **added** to the balance and the amount you pay off is **subtracted** from it.*

$u_{n+1} = u_n + 0.015u_n - 50 = 1.015u_n - 50$, with $u_0 = 500$

*Remember to give the **initial condition**.*

### Example 2

The deer population of a county was observed to be 1200 in a given year. The population is modelled to increase at a rate of 15% each year. Let $d_n$ be the population of deer $n$ years later. Explain why the deer population is modelled by the recurrence relation

$$d_n = 1.15d_{n-1}, \text{ with } d_0 = 1200$$

After $n - 1$ years the population is $d_{n-1}$

This is increased by 15%, so the population after $n$ years is $d_{n-1} + 0.15d_{n-1} = 1.15d_{n-1}$

The initial population is 1200, so $d_0 = 1200$

*Explain the recurrence relation in the context of the question.*

## Example 3

A

A population of bacteria has initial size 200. After one hour, the population has reached 220. The population grows in such a way that the rate of growth doubles each hour. Write a recurrence relation to describe the number of bacteria, $b_n$, after $n$ hours.

$b_0 = 200$ and $b_1 = 220$
The increase from time $n - 1$ to $n$ is $b_n - b_{n-1}$, and the increase from time $n - 2$ to $n - 1$ is $b_{n-1} - b_{n-2}$
So $b_n - b_{n-1} = 2(b_{n-1} - b_{n-2})$
$b_n = 3b_{n-1} - 2b_{n-2}$, with $b_0 = 200$, $b_1 = 200$

The **rate** of growth doubles each hour. So the increase from time $n - 1$ to $n$ will be double the increase from time $n - 2$ to $n - 1$.

**Notation** This is an example of a **second-order recurrence relation**, as $b_{n+1}$ is given in terms of **two previous terms**, $b_{n-1}$ and $b_{n-2}$. You need **two initial conditions** to define the sequence, given here in terms of $b_0$ and $b_1$. → Section 7.3

If you know the general term of a sequence in the form $u_n = f(n)$, you can verify that it satisfies a given recurrence relation by substitution.

## Example 4

A sequence has the general term $u_n = 3n - 1$. Verify that the sequence satisfies the recurrence relation $u_n = 3 + u_{n-1}$.

$u_n = 3n - 1$, so $u_{n-1} = 3(n - 1) - 1 = 3n - 4$
Substituting into the RHS of the recurrence relation,
$3 + u_{n-1} = 3 + (3n - 4)$
$= 3n - 1$
$= u_n$ as required

**Watch out** This is not the only general term that satisfies this recurrence relation. Any general term of the form $u_n = 3n + k$, where $k$ is a constant, will also satisfy the recurrence relation.

## Example 5

A sequence has the general term $u_n = 2 \times 3^{n-1}$. Verify that the sequence satisfies the recurrence relation $u_n = 3u_{n-1}$.

$u_{n-1} = 2 \times 3^{(n-1)-1} = 2 \times 3^{n-2}$
Substituting into the RHS of the recurrence relation,
$3u_{n-1} = 3(2 \times 3^{n-2}) = 2 \times 3^{n-1} = u_n$

**Notation** $u_n = 3u_{n-1}$ is the **recursive form** of the sequence. $u_n = 2 \times 3^{n-1}$ is the **solution**, or the **closed form** of the sequence. It is also sometimes called the **explicit form** of the sequence. → Section 7.2

## Exercise 7A

**1** The value of an endowment policy increases at a rate of 5% per annum. The initial value of the policy is £7000.

**a** Write down a recurrence relation for the value of the policy after $n$ years.

**b** Calculate the value of the policy after 4 years.

**Hint** Remember to include an initial condition in your answer to part **a**.

**2** A patient is injected with 156 ml of a drug. Every 8 hours, 22% of the drug passes out of his bloodstream, and a further 25 ml dose of the drug is administered. After $8n$ hours, the amount of the drug in the patient's bloodstream is $d_n$ ml.

**a** Find an expression for $d_n$ in terms of $d_{n-1}$, and write down the value of $d_0$.

**b** Calculate, to the nearest millilitre, the amount of drug in the patient's bloodstream after 24 hours.

(E) **3** Kandace takes out a personal loan of £5000 to buy a car. The interest rate on the loan is 0.5% per month. Interest is calculated and added to the loan balance at the end of each month. At the end of each month, Kandace makes a monthly payment of £200, which is deducted from the balance of the loan. The balance in pounds at the end of the $n$th month is given by $b_n$. Explain why $b_n = kb_{n-1} - 200$, with $b_0 = 5000$, and find the value of the constant $k$. **(3 marks)**

(E) **4** At the time a census is taken, the population of a country is 12.5 million. The annual birth rate is 4% and the annual death rate is 3%. In addition, each year there is a net migration of 50 000 new immigrants into the country.
Write a recurrence relation for the population of the country $n$ years after the census, $P_n$. **(3 marks)**

**5** A sequence has general term $u_n = 5n + 2$. Verify that the sequence satisfies the recurrence relation $u_n = u_{n-1} + 5$.

**6** A sequence has general term $u_n = 6 \times 2^n + 1$. Verify that the sequence satisfies the recurrence relation $u_n = 2u_{n-1} - 1$.

(P) **7** Consider the sequence given by $u_n = \sum_{i=1}^{n}(2i - 1)$

**a** Write down the first 4 terms of the sequence.

**b** Explain why the recurrence relation associated with this sequence is

$$u_{n+1} = u_n + 2n + 1,\ n \geqslant 1$$

**c** Verify that $u_n = n^2$ is a solution to this recurrence relation.

(E/P) **8** In January 2010, a small oil company produced 2000 barrels of oil and sold 1800 barrels of oil. Any remaining oil was stockpiled. From January 2010 onwards, the company increased its sales by 20 barrels per month, and increased its oil production by 1% each month.

**a** Find an expression for:

**i** the number of barrels produced by the well in the $n$th month

**ii** the number of barrels sold in the $n$th month. **(4 marks)**

At the beginning of January 2010, the oil company had no stockpiled oil.

**b** Find a recurrence relation for the total number of stockpiled barrels, $s_n$, at the end of the $n$th month. **(3 marks)**

E/P **9** There are $n$ people at a gathering. Each person shakes hands with everybody else exactly once. Let $h(n)$ be the number of handshakes that occur.

**a** Explain why $h(1) = 0$. **(1 mark)**

**b** Find a recurrence relation for $h(n + 1)$ in terms of $h(n)$. **(2 marks)**

A **10** Generate the first six terms of each of the following sequences:

**a** $u_n = 2u_{n-1} + 3u_{n-2}$, with $u_0 = 1$ and $u_1 = 1$

**b** $u_n = u_{n-1} - 2u_{n-2}$, with $u_0 = 1$ and $u_1 = 1$

**c** $u_n = u_{n-1} + u_{n-2} + 2n$, with $u_0 = 1$ and $u_1 = 1$

E **11** Assume that growth in a bacterial population has the following properties:

- At the beginning of every hour, each bacterium that lived in the previous hour divides into two new bacteria. During the hour, all bacteria that have lived for two hours die.
- At the beginning of the first hour, the population consists of 100 bacteria.

At the end of the $n$th hour there are $B_n$ bacteria in the population.
Find a recurrence relation for $B_n$. **(3 marks)**

P **12** A sequence has $n$th term $u_n = (2 - n)2^{n+1}$
Verify that the sequence satisfies the recurrence relation $u_n = 4(u_{n-1} - u_{n-2})$.

E/P **13** A battery-operated kangaroo is able to make two kinds of jumps: small jumps of length 10 cm or large jumps of length 20 cm. The number of different ways in which the kangaroo can cover a distance of $10n$ cm is denoted by $J_n$.

**a** By writing down all possible combinations of jumps for a distance of 40 cm, show that $J_4 = 5$. **(2 marks)**

**b** Find a recurrence relation for $J_n$, stating the initial conditions. **(3 marks)**

**c** How many different ways can this kangaroo cover a distance of 80 cm? **(1 mark)**

E/P **14** A female rabbit is modelled as producing 2 surviving female offspring in its first year of life, and 6 in each subsequent year. A population initially has 4 female rabbits, all of whom are more than 1 year old.

**a** If $F_n$ is the number of female rabbits in the population after $n$ years, explain why $F_n$ is modelled by the recurrence relation

$$F_n = 3F_{n-1} + 4F_{n-2}, \text{ with } F_0 = 4 \text{ and } F_1 = 28$$ **(3 marks)**

**b** Suggest a criticism of this model. **(1 mark)**

A E/P **15** Binary strings consist of 1s and 0s.
There are 5 different binary strings of length 3 which **do not** contain consecutive 1s:

000, 001, 010, 100, 101

**Hint** For example, 011 is not allowable because it contains consecutive 1s.

Let $b_n$ represent the number binary strings of length $n$ with no consecutive 1s.

**a** Find $b_1$ and $b_2$. **(1 mark)**

**b** Explain why $b_n$ satisfies the recurrence relation $b_n = b_{n-1} + b_{n-2}$. **(3 marks)**

**c** Hence find $b_7$. **(1 mark)**

## 7.2 Solving first-order recurrence relations

You need to be able to **solve** recurrence relations. This means finding a **closed form** for the terms in the sequence in the form $u_n = f(n)$.

- **The order of a recurrence relation is the difference between the highest and lowest subscripts in the relation.**
- **A first-order recurrence relation is one in which $u_n$ can be given as a function of $n$ and $u_{n-1}$ only.**

Examples of first-order recurrence relations are:

$u_n = 2u_{n-1} + n$
$a_n = (n + 1)a_{n-1}$
$P_{n+1} = 5P_n + 2n^2$

Here the subscripts given are $n + 1$ and $n$. This is still a first-order recurrence relation because the difference between them is 1.

In this section you will learn how to solve first-order **linear** recurrence relations.

- **A first-order linear recurrence relation can be written in the form $u_n = au_{n-1} + g(n)$, where $a$ is a real constant.**
  - **If $g(n) = 0$, then the equation is homogeneous.**

You can sometimes find solutions to recurrence relations using a technique called **back substitution**.

### Example 6

Find a closed form for the sequence

$a_n = 5a_{n-1}$, $n > 0$, with $a_0 = 1$

$a_n = 5a_{n-1}$
$= 5 \times 5a_{n-2} = 5^2a_{n-2}$
$= 5^2 \times 5a_{n-3} = 5^3a_{n-3}$
$= \ldots = 5^n a_0$.
$a_0 = 1$, so the closed form of this sequence is $a_n = 5^n$

$a_{n-1} = 5a_{n-2}$. Substitute this into the expression for $a_n$.

This is the geometric sequence 1, 5, 25, 125 …

The recurrence relation in the example above is an example of a **homogeneous** recurrence relation. A first-order homogeneous linear recurrence relation can be written in the form $u_n = au_{n-1}$.

Using back substitution,

$$\begin{aligned} u_n &= au_{n-1} \\ &= a \times au_{n-2} = a^2u_{n-2} \\ &= a^2 \times au_{n-3} = a^3u_{n-3} \\ &\vdots \\ &= a^{n-1}u_1 \\ &= a^n u_0 \end{aligned}$$

- **The solution to the first-order homogeneous linear recurrence relation $u_n = au_{n-1}$ is given by $u_n = u_0a^n$ or $u_n = u_1a^{n-1}$.**

You can write down these solutions in your exam. You do not need to use back substitution.

## Example 7

Solve the recurrence relation $a_n = 2a_{n-1}$, $n \geqslant 1$, with $a_0 = 3$.

$$\begin{aligned} a_n &= a_0(2^n) \\ &= 3(2^n) \end{aligned}$$

This is a homogeneous linear first-order recurrence relation, so use the rule given above to write down the general solution.

It is useful to think of a **general solution** to the recurrence relation $u_n = au_{n-1}$ in the form $u_n = ca^n$, where $c$ is an arbitrary constant. You can then use the initial conditions to find the value of $c$. This will give you a **particular solution**.

**Links** The process of finding general solutions (with arbitrary constants), and then using initial conditions to find particular solutions, is very similar to the process of solving a differential equation. You can think of a recurrence relation as a discrete version of a differential equation. ← Pure Year 2, Section 11.10

## Example 8

Solve the recurrence relation $a_n = -3a_{n-1}$, $n \geqslant 1$, with $a_1 = 6$.

**Method 1**

$$\begin{aligned} a_n &= a_1(-3)^{n-1} \\ &= 6(-3)^{n-1} \end{aligned}$$

Use the form of the solution $u_n = u_1r^{n-1}$.

**Method 2**

General solution is $a_n = c(-3)^n$

$a_1 = 6 \Rightarrow 6 = c(-3)^1 \Rightarrow c = -2$

Therefore, the particular solution is $a_n = -2(-3)^n$.

Write a general solution with an arbitrary constant, then use the initial condition to find the value of the constant.

**Problem-solving**

The two solutions are equivalent:

$$-2(-3)^n = -2(-3)(-3)^{n-1} = 6(-3)^{n-1}$$

You can find solutions to some non-homogeneous linear recurrence relations using back-substitution.

## Example 9

Find a solution to the recurrence relation $u_n = u_{n-1} + n$, $n \geqslant 1$, with $u_0 = 0$.

Using iteration,

$u_n = u_{n-1} + n$

$= (u_{n-2} + (n-1)) + n$

$= (u_{n-3} + (n-2)) + (n-1) + n$

$\vdots$

$= (u_1 + 2) + 3 + 4 + \ldots + n$

$= u_0 + 1 + 2 + \ldots + n$

$= u_0 + \sum_{r=1}^{n} r$

$= \frac{n(n+1)}{2}$

Therefore, the closed form for this recurrence relation is $u_n = \frac{n(n+1)}{2}$

Replace $n$ by $n - 1$ in $u_n = u_{n-1} + n$, then substitute. Repeat this process for $n - 1$, $n - 2$ and so on.

$u_0 = 0$ and the sum of the first $n$ integers is $\frac{n(n+1)}{2}$

← Core Pure Book 1, Section 3.1

You can apply this method to any recurrence relation of the form $u_n = u_{n-1} + \text{g}(n)$:

$u_n = u_{n-1} + \text{g}(n)$

$= (u_{n-2} + \text{g}(n-1)) + \text{g}(n)$

$= ((u_{n-3} + \text{g}(n-2)) + \text{g}(n-1)) + \text{g}(n)$

$\vdots$

$= u_0 + \sum_{r=1}^{n} \text{g}(r)$

- **The solution to the first-order non-homogeneous linear recurrence relation $u_n = u_{n-1} + \text{g}(n)$ is given by $u_n = u_0 + \sum_{r=1}^{n} \text{g}(r)$.**

**Watch out** If $u_1$ is given instead of $u_0$, the solution would be $u_1 + \sum_{r=2}^{n} \text{g}(r)$

## Example 10

Solve the following recurrence relations.

**a** $u_n = u_{n-1} + 2n + 1$, $n \geqslant 0$, with $u_0 = 7$

**b** $u_n = u_{n-1} + 5^n$, $n \in \mathbb{N}$, with $u_1 = 3$

**a** $u_n = u_0 + \sum_{r=1}^{n} \text{g}(r)$

$= 7 + \sum_{r=1}^{n} (2r + 1)$

$= 7 + 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$= 7 + n(n+1) + n$

$= n^2 + 2n + 7$

Use the formula for the solution to a recurrence relation of the form $u_n = u_{n-1} + \text{g}(n)$.

Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} 1$.

← Core Pure Book 1, Section 3.1

**b** Method 1

$u_0 = u_1 - 5^1 = -2$

$u_n = u_0 + \sum_{r=1}^{n} g(r)$

$= -2 + \sum_{r=1}^{n} 5^r$

$= -2 + \frac{5(1 - 5^n)}{1 - 5}$

Method 2

$u_n = u_1 + \sum_{r=2}^{n} g(r) = 3 + \sum_{r=2}^{n} 5^r$

$= 3 + \frac{5^2(1 - 5^{n-1})}{1 - 5}$

$= -2 - \frac{5}{4} + \frac{5^{n+1}}{4}$

$= \frac{1}{4}(5^{n+1}) - \frac{13}{4}$

The initial condition is given in terms of $u_1$, so you need to find an expression for $u_0$ before you can use the formula.

$\sum_{r=1}^{n} 5^r$ is a geometric series with $n$ terms, first term 5 and common ratio 5. ← **Pure Year 2, Chapter 3**

If you need to solve a recurrence relation of the form $u_n = au_{n-1} + g(n)$, where $a \neq 1$, back substitution gets more complicated. You can solve non-homogeneous recurrence relations of this form by first considering the general solution to the corresponding homogeneous recurrence relation, $u_n = au_{n-1}$. This general solution is called the **complementary function (C.F.)**. You then need to add a **particular solution (P.S.)** to the recurrence relation.

**Links** The particular solution plays a similar role to the particular integral which is used when solving a second-order linear differential equation. → **Core Pure Book 2, Section 7.3**

- **When solving a recurrence relation of the form $u_n = au_{n-1} + g(n)$, the form of the particular solution will depend on g($n$):**

| Form of g($n$) | Form of particular solution |
|---|---|
| $p$ with $a \neq 1$ | $\lambda$ |
| $pn + q$ with $a \neq 1$ | $\lambda n + \mu$ |
| $kp^n$ with $p \neq a$ | $\lambda p^n$ |
| $ka^n$ | $\lambda n a^n$ |

**Watch out** This particular solution will satisfy the whole recurrence relation but will not necessarily satisfy the initial condition.

- **To solve the recurrence relation $u_n = au_{n-1} + g(n)$,**
  - **Find the complementary function (C.F.), which is the general solution to the associated homogeneous recurrence relation $u_n = au_{n-1}$.**
  - **Choose an appropriate form for a particular solution (P.S.) then substitute into the original recurrence relation to find the values of any coefficients.**
  - **The general solution is $u_n$ = C.F. + P.S. = $ca^n$ + P.S.**
  - **Use the initial condition to find the value of the arbitrary constant.**

**Watch out** The term **particular solution** is also sometimes used to refer to the final solution given the initial condition. Both versions satisfy the recurrence relation but only the final solution satisfies the initial condition.

You can use this method when $a = 1$, but in this case the complementary function is a constant, so you need to find a particular solution with no constant terms. You can do this by multiplying the particular solution by $n$:

| Form of g($n$) | Form of particular solution |
|---|---|
| $p$ with $a = 1$ | $\lambda n$ |
| $pn + q$ with $a = 1$ | $\lambda n^2 + \mu n$ |

**Note** For recurrence relations of the form $u_n = u_{n-1} + p$ or $u_n = u_{n-1} + pn + q$, it is usually easier to use the summation formula given on page 221.

## Example 11

Solve the recurrence relation $u_n = 3u_{n-1} + 2n$, $n \in \mathbb{Z}^+$, with $u_1 = 3$.

Associated homogeneous recurrence relation is $u_n = 3u_{n-1}$

Complementary function: $u_n = c(3^n)$

Find the general solution to the associated homogeneous recurrence relation. This is the **complementary function (C.F.)**.

Particular solution: $u_n = \lambda n + \mu$

g($n$) is of the form $pn + q$, so look for a particular solution of the form $\lambda n + \mu$. You need to include the constant term even though g($n$) does not have a constant term.

$$u_n = 3u_{n-1} + 2n$$
$$\lambda n + \mu = 3(\lambda(n-1) + \mu) + 2n$$
$$\lambda n + \mu = 3\lambda n - 3\lambda + 3\mu + 2n$$
$$0 = (2\lambda + 2)n + (2\mu - 3\lambda)$$

Substitute $u_n = \lambda n + \mu$ and $u_{n-1} = \lambda(n-1) + \mu$ into the full recurrence relation.

$\Rightarrow 2\lambda + 2 = 0$ and $2\mu - 3\lambda = 0$

$\Rightarrow \lambda = -1, \mu = -\frac{3}{2}$

**Problem-solving** Simplify, and group together coefficients of $n$ and constant coefficients. Since the values of $\lambda$ and $\mu$ must satisfy the recurrence relation for **any value of $n$**, you can consider it as an identity. This means that you can equate coefficients with the same power of $n$ on both sides.

Solve these two equations simultaneously.

So a particular solution to the recurrence relation is

$$u_n = -n - \frac{3}{2}$$

The general solution is $u_n = c(3^n) - n - \frac{3}{2}$

General solution = C.F. + P.S.

Since $u_1 = 3$,

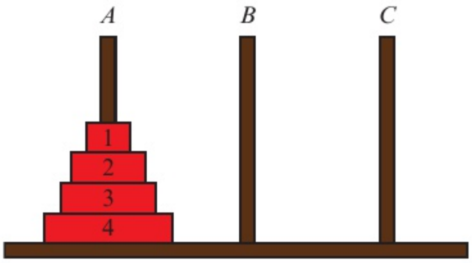
$3 = c(3^1) - 1 - \frac{3}{2} \Rightarrow c = \frac{11}{6}$

Use the initial condition $u_1 = 3$ to find the value of $c$.

The solution is $u_n = \frac{11}{6}(3^n) - n - \frac{3}{2}$

## Example 12

The Tower of Hanoi puzzle involves transferring a pile of different sized disks from one peg to another using an intermediate peg.




The rules are as follows:

- Only one disk at a time can be moved.
- A disk can only be moved if it is the top disk on a pile.
- A larger disk can never be placed on a smaller one.

**a** Find the minimum number of moves needed to transfer two disks from one peg to another.

**b** Show that three disks can be transferred from one peg to another in 7 moves.

**c** Explain why the minimum number of moves, $d_n$, needed to transfer $n$ disks from one peg to another satisfies the recurrence relation $d_n = 2d_{n-1} + 1$, with $d_1 = 1$.

**d** Solve this recurrence relation for $d_n$.

**e** Hence determine the minimum number of moves needed to transfer 15 disks from one peg to another.

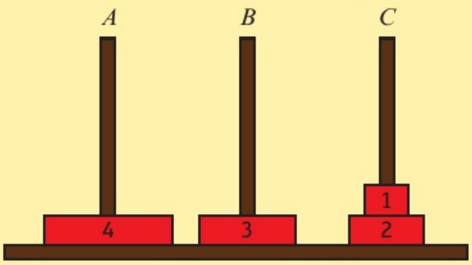
**a** 3 moves

For example, move disk 1 from $A$ to $B$, disk 2 from $A$ to $C$, then disk 1 from $B$ to $C$.

**b**

| Move number | Disk | From | To |
|---|---|---|---|
| 1 | 1 | $A$ | $B$ |
| 2 | 2 | $A$ | $C$ |
| 3 | 1 | $B$ | $C$ |
| 4 | 3 | $A$ | $B$ |
| 5 | 1 | $C$ | $A$ |
| 6 | 2 | $C$ | $B$ |
| 7 | 1 | $A$ | $B$ |

After move 4 the disks are arranged as follows:




**c** Before you can move the largest disk (disk $n$), you must have transferred all the other disks to a single peg, say $C$. This requires $d_{n-1}$ moves. You then move disk $n$ in 1 move, to peg $B$. Finally, transfer the other disks to be on top of disk $n$, on peg $B$. This requires a further $d_{n-1}$ moves. So the total number of moves is

$$d_n = d_{n-1} + 1 + d_{n-1} = 2d_{n-1} + 1$$

One disk can be transferred in one move so $d_1 = 1$.

Make sure you explain why the initial condition is true as well.

**Online** Play the Tower of Hanoi using Geogebra.

**d** Associated homogeneous recurrence relation: $d_n = 2d_{n-1}$
Complementary function: $d_n = c(2^n)$
Particular solution: $d_n = \lambda$
$d_n = 2d_{n-1} + 1$
$\lambda = 2\lambda + 1$
$\lambda = -1$
So a particular solution to the recurrence relation is $d_n = -1$.
The general solution is $d_n = c(2^n) - 1$
Since $d_1 = 1$,
$1 = c(2^1) - 1 \Rightarrow c = 1$
The solution is $d_n = 2^n - 1$

**e** $d_{15} = 2^{15} - 1 = 32767$

Find the general solution to the associated homogeneous recurrence relation.

The recurrence relation is of the form $u_n = au_{n-1} + g(n)$ with $a \neq 1$ and $g(n) = p$, a constant, so try a particular solution of the form $u_n = \lambda$.

Substitute $d_n = \lambda$ and $d_{n-1} = \lambda$ into the full recurrence relation and solve to find $\lambda$.

General solution = C.F. + P.S.

Use the initial condition $d_1 = 1$ to find the value of the arbitrary constant, $c$.

## Exercise 7B

**1** Find the solution to each of the following recurrence relations.

**a** $u_n = 2u_{n-1}$, with $u_0 = 5$

**b** $b_n = \frac{5}{2}b_{n-1}$, with $b_1 = 4$

**c** $d_n = -\frac{11}{10}d_{n-1}$, with $d_1 = 10$

**d** $x_{n+1} = -3x_n$, with $x_0 = 2$

**2** Find a closed form for the sequences defined by the following recurrence relations.

**a** $u_n = u_{n-1} + 3$, with $u_0 = 5$

**b** $x_n = x_{n-1} + n$, with $x_0 = 2$

**c** $y_n = y_{n-1} + n^2 - 2$, with $y_0 = 3$

**d** $s_{n+1} = s_n + 2n - 1$, with $s_0 = 1$

**Watch out** In part **d**, the summation indices are slightly different, so this recurrence relation is not in the form $u_n = u_{n-1} + g(n)$. If you substitute $n$ for $n - 1$ throughout the recurrence relation you can use the formula $u_n = u_0 + \sum_{r=1}^{n} g(r)$

**3** Solve each of the following recurrence relations.

**a** $a_n = 2a_{n-1} + 1$, with $a_1 = 1$

**b** $u_n = -u_{n-1} + 2$, with $u_1 = 3$

**c** $h_n = 3h_{n-1} + 5$, with $h_0 = 1$

**d** $b_n = -2b_{n-1} + 6$, with $b_1 = 3$

**Hint** In each case, use a constant particular solution of the form $\lambda$.

(E) **4** In a league of $n$ football teams, each team plays every other team exactly once. In total, $g_n$ matches are played.

**a** Explain why $g_n = g_{n-1} + n - 1$, and write down a suitable initial condition for this recurrence relation. **(3 marks)**

**b** By solving your recurrence relation, show that $g_n = \dfrac{n(n-1)}{2}$ **(4 marks)**

**5** **a** Find the general solution to the recurrence relation $u_n = 4u_{n-1} - 1$, $n \geqslant 2$

**b** Hence or otherwise find the particular solution given that:

**i** $u_1 = 3$ **ii** $u_1 = 0$ **iii** $u_1 = 200$

E/P **6** **a** Find the general solution to the recurrence relation $u_n = 3u_{n-1} + n$, $n > 1$. **(3 marks)**

**b** Given that $u_1 = 5$, find the particular solution to this recurrence relation. **(1 mark)**

E **7** A sequence is defined by the recurrence relation $u_{n+1} = 0.6u_n + 4$, with $u_0 = 7$.

**a** Find $u_3$. **(1 mark)**

**b** Find a closed form for the recurrence relation. **(3 marks)**

**c** Find the smallest value of $n$ for which $u_n > 9.9$. **(1 mark)**

E/P **8** The deer population in a forest is estimated to drop by 5% each year.
Each year, 20 additional deer are introduced to the forest.
The initial deer population is 200, and the population after $n$ years is given by $D_n$.

**a** Write down a recurrence relation for $D_n$. **(3 marks)**

**b** By solving your recurrence relation, find an expression for $D_n$ in terms of $n$. **(3 marks)**

**c** Describe the behaviour of the deer population in the long term. **(1 mark)**

E **9** Solve the recurrence relation $u_n - 4u_{n-1} + 3 = 0$, with $u_0 = 7$. **(3 marks)**

E **10** A sequence of numbers satisfies the recurrence relation

$$u_n = u_{n-1} + 2^n,\ n \geqslant 2, \text{ with } u_1 = 5$$

Find a closed form for $u_n$. **(3 marks)**

E **11** Solve the recurrence relation $u_n = 4u_{n-1} + 2n$, with $u_0 = 7$. **(4 marks)**

E/P **12** A sequence satisfies the recurrence relation $u_n = 2u_{n-1} - 1005$, with $u_0 = 1000$.

**a** Solve the recurrence relation to find a closed form for $u_n$. **(4 marks)**

**b** Hence, or otherwise, find the first negative term in the sequence. **(3 marks)**

E/P **13** **a** Find the general solution to the recurrence relation $u_n = 2u_{n-1} - 2^n$, $n \geqslant 2$. **(4 marks)**

**b** Find the particular solution to this recurrence relation given that $u_1 = 3$. **(1 mark)**

E/P **14** A sequence is defined by the recurrence relation $u_n = ku_{n-1} + 1$, $k \neq 1$, with $u_0 = 0$.

**a** Find the value of $u_1$, $u_2$, and $u_3$ in terms of $k$. **(2 marks)**

**b** Find a closed form for this sequence. **(3 marks)**

**c** Describe the behaviour of the sequence as $n$ gets very large in the cases when:

**i** $k > 1$ **ii** $-1 < k < 1$ **iii** $k = -1$ **iv** $k < -1$ **(4 marks)**

E/P **15** A sequence is defined by the recurrence relation

$$a_n = a_{n-1} + 6n + 1,\ n \in \mathbb{Z}^+, \text{ with } a_0 = 2$$

**a** Find $\sum_{r=1}^{n}(6r + 1)$ **(2 marks)**

**b** Hence, or otherwise, find a closed form for this sequence. **(3 marks)**

**c** Given that $a_n = 561$, find the value of $n$. **(2 marks)**

**E/P** **16** **a** Solve the recurrence relation $u_n = u_{n-1} - 6n^2$, with $u_0 = 89$. **(3 marks)**

**b** Hence, or otherwise, find the first negative term of the sequence. **(2 marks)**

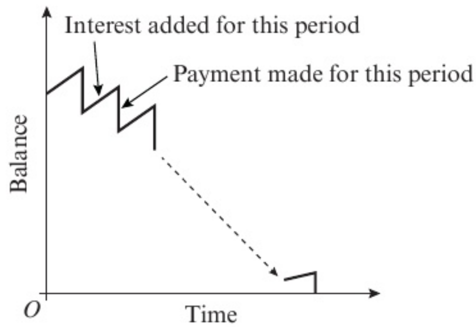
**c** Explain why every term of the sequence is an odd number. **(2 marks)**

**E/P** **17** **a** Solve the recurrence relation $u_n = u_{n-1} - 2n$, with $u_0 = 3$. **(2 marks)**

**b** Show that $-103$ is not a term of the sequence. **(2 marks)**

**c** Given that $u_k = -459$, find the value of $k$. **(2 marks)**

**E/P** **18** Alison borrows £2000 on her credit card. She intends to pay it back by making 18 monthly payments. At the end of each month, interest of 1.5% is added to the loan balance, and Alison's monthly payment of £$P$ is deducted from the loan balance. The graph illustrates how the balance of the loan will change over time.




**a** Write a recurrence relation for the balance of the loan at the end of $n$ months. **(3 marks)**

**b** Find a solution to your recurrence relation, giving your answer in terms of $P$. **(3 marks)**

Alison wants the balance of the loan to be zero after she makes her 18th payment.

**c** Find the value of $P$ that will make this the case. **(3 marks)**

**Challenge**

A **restricted** Tower of Hanoi problem requires a player to move a pile of disks of different sizes from peg $A$ to peg $C$. The rules are as follows:

- Only one disk at a time can be moved.
- A disk can only be moved if it is the top disk on a pile.
- A larger disk can never be placed on a smaller one.
- **Disks can only be moved a distance of one peg at a time.**

Let $H_n$ be the minimum number of moves needed to transfer $n$ discs from peg $A$ to peg $C$.

**a** Explain why $H_1 = 2$.

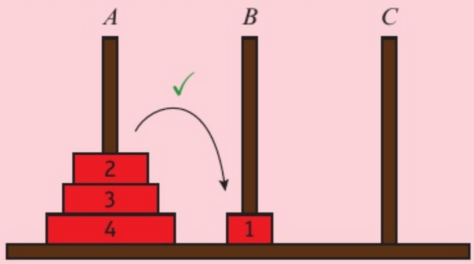
**b** Show that 2 disks can be moved from peg $A$ to peg $C$ in 8 moves.

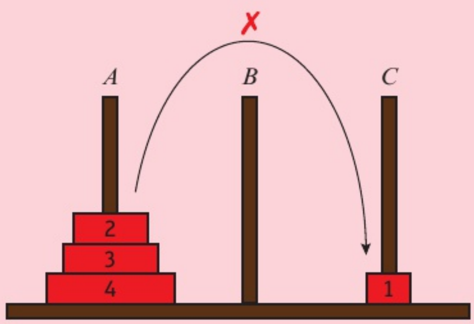
**c** Explain why $H_n$ satisfies a recurrence relation of the form $H_n = aH_{n-1} + b$, and determine the values of $a$ and $b$.

**d** **i** Solve this recurrence relation for $H_n$.

**ii** Hence determine the minimum number of moves needed to transfer 10 disks from peg $A$ to peg $C$.

**Watch out** The fourth rule means that moves between pegs $A$ and $B$, and pegs $B$ and $C$ are allowed, but direct moves between pegs $A$ and $C$ are not.



## 7.3 Solving second-order recurrence relations

A

In this section you will learn how to solve second-order linear recurrence relations.

- **A second-order linear recurrence relation can be written in the form $u_n = au_{n-1} + bu_{n-2} + \mathrm{g}(n)$, where $a$ and $b$ are real constants.**
  - **If $\mathrm{g}(n) = 0$, then the equation is homogeneous.**

### Example 13

Consider the recurrence relation $u_n = 2u_{n-1} - u_{n-2}$, $n \geqslant 2$.

Verify that the following particular solutions satisfy this recurrence relation.

**a** $u_n = 3n$ **b** $u_n = 5$ **c** $u_n = 3n + 5$

**a** $u_n = 3n$, $u_{n-1} = 3(n - 1) = 3n - 3$
$u_{n-2} = 3(n - 2) = 3n - 6$
$2u_{n-1} - u_{n-2} = 2(3n - 3) - (3n - 6) = 3n = u_n$
So $u_n = 3n$ satisfies the recurrence relation.

Find expressions for $u_{n-1}$ and $u_{n-2}$ and substitute them into the RHS of the recurrence relation.

**b** $u_n = 5$, $u_{n-1} = 5$, $u_{n-2} = 5$
$2u_{n-1} - u_{n-2} = 2 \times 5 - 5 = 5 = u_n$
So $u_n = 5$ satisfies the recurrence relation.

**c** $u_n = 3n + 5$, $u_{n-1} = 3(n - 1) + 5 = 3n + 2$
$u_{n-2} = 3(n - 2) + 5 = 3n - 1$
$2u_{n-1} - u_{n-2} = 2(3n + 2) - (3n - 1) = 3n + 5 = u_n$
So $u_n = 3n + 5$ satisfies the recurrence relation.

- **If $u_n = \mathrm{F}(n)$ and $u_n = \mathrm{G}(n)$ are particular solutions to a linear recurrence relation, then $u_n = a\mathrm{F}(n) + b\mathrm{G}(n)$, where $a$ and $b$ are constants, is also a solution.**

You can solve a **second-order homogeneous linear** recurrence relation by looking for solutions of the form $u_n = Ar^n$, where $A$ is an arbitrary non-zero constant.

Suppose that $u_n = Ar^n$ is a solution to the recurrence relation $u_n = au_{n-1} + bu_{n-2}$.

Then $Ar^n = Aar^{n-1} + Abr^{n-2}$
$\Rightarrow r^2 - ar - b = 0$

Multiply both sides by $r^{2-n}$ and simplify.

This quadratic equation is called the **auxiliary equation** of the recurrence relation. $u_n = Ar^n$ is a solution to the recurrence relation if and only if $r$ is a root of this equation.

**Notation** The auxiliary equation is sometimes called the **characteristic equation**.

However, a second-order recurrence relation requires **two initial conditions** to fully define the sequence. As such, the general solution to a second-order recurrence relation requires **two arbitrary constants**. You can formulate a general solution with two arbitrary constants by adding multiples of two different solutions.

A

- **You can find a general solution to a second-order homogeneous linear recurrence relation, $u_n = au_{n-1} + bu_{n-2}$, by considering the auxiliary equation**

  $$r^2 - ar - b = 0$$

  **You need to consider three different cases:**

  - **Case 1: Distinct real roots**
    **If the auxiliary equation has distinct real roots $\alpha$ and $\beta$, then the general solution will have the form $u_n = A\alpha^n + B\beta^n$ where $A$ and $B$ are arbitrary constants.**

  - **Case 2: Repeated root**
    **If the auxiliary equation has a repeated real root $\alpha$, then the general solution will have the form $u_n = (A + Bn)\alpha^n$ where $A$ and $B$ are arbitrary constants.**

  - **Case 3: Complex roots**
    **If the auxiliary equation has two complex roots $\alpha = re^{i\theta}$ and $\beta = re^{-i\theta}$, then the general solution will have the form $u_n = r^n(A\cos n\theta + B\sin n\theta)$, or $u_n = A\alpha^n + B\beta^n$, where $A$ and $B$ are arbitrary constants.**

**Links** These three cases are similar to the cases you consider when solving a differential equation of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$.

→ Core Pure Book 2, Section 7.2

## Example 14

**a** Find a general solution to the recurrence relation

$u_n = 2u_{n-1} + 8u_{n-2}, n \geqslant 2$

**b** Verify that your general solution from part **a** satisfies the recurrence relation.

**c** Given than $u_0 = 4$ and $u_1 = 10$, find a particular solution.

**a** $r^2 - 2r - 8 = 0$

$(r - 4)(r + 2) = 0$

$\Rightarrow r = 4$ or $r = -2$

Write down the auxiliary equation and solve it.

General solution is $u_n = A(4^n) + B(-2)^n$

The auxiliary equation has two distinct real roots, so the general solution has the form $u_n = A\alpha^n + B\beta^n$.

**b** $2u_{n-1} + 8u_{n-2}$

$= 2(A(4^{n-1}) + B(-2)^{n-1}) + 8(A(4^{n-2}) + B(-2)^{n-2})$

$= 2A(4^{n-1}) + 2B(-2)^{n-1} + 8A(4^{n-2}) + 8B(-2)^{n-2}$

$= 2A(4^{n-1}) + 2B(-2)^{n-1} + 2A(4^{n-1}) - 4B(-2)^{n-1}$

$= 4A(4^{n-1}) - 2B(-2)^{n-1}$

$= A(4^n) + B(-2)^n$

$= u_n$

Substitute your general solution into the RHS of the recurrence relation, then take out factors of 4 and −2 to write the expression in terms of multiples of $4^n$ and $(-2)^n$.

**c** $u_0 = 4 \Rightarrow A(4^0) + B(-2)^0 = 4 \Rightarrow A + B = 4$

$u_1 = 10 \Rightarrow A(4^1) + B(-2)^1 = 10 \Rightarrow 4A - 2B = 10$

Solving simultaneously, $A = 3$ and $B = 1$

So the solution is $u_n = 3(4^n) + (-2)^n$

Use the initial conditions to write two simultaneous equations in $A$ and $B$. Solve these to find the values of the arbitrary constants.

## Example 15

A

Solve the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$, with $a_1 = 5$ and $a_2 = 3$.

$r^2 - 3r + 2 = 0$
$(r - 1)(r - 2) = 0$
$\Rightarrow r = 1$ or $r = 2$

Write down the auxiliary equation and solve it.

So the general solution is $a_n = A(1^n) + B(2^n)$
$= A + B(2^n)$

One of the roots is 1, so one of the terms in the general solution will be constant.

$a_1 = A + 2B = 5$
$a_2 = A + 4B = 3$
$\Rightarrow A = 7, B = -1$

Use the initial conditions to find the arbitrary constants.

So the solution is $a_n = 7 - 2^n$.

**Problem-solving**

You can check your answer by generating the first few terms of the sequence using the solution and the original recurrence relation. The first 5 terms here are 5, 3, −1, −9, and −25.

## Example 16

Solve the recurrence relation $u_n = 4u_{n-1} - 4u_{n-2}$, with $u_0 = 1$ and $u_1 = 1$.

$r^2 - 4r + 4 = 0$
$(r - 2)^2 = 0$
$\Rightarrow r = 2$
So the general solution is $u_n = (A + Bn)2^n$

The auxiliary equation has one repeated root, so the general solution is of the form $u_n = (A + Bn)\alpha^n$.

$u_0 = 1 \Rightarrow A = 1$
$u_1 = 1 \Rightarrow 2(A + B) = 1$
$2 + 2B = 1$
$B = -\frac{1}{2}$

Use the initial conditions to form two equations and solve these to find $A$ and $B$.

The solution is $u_n = (1 - \frac{1}{2}n)2^n$.

You could also write this as $u_n = 2^n - n2^{n-1}$.

## Example 17

**a** Find the general solution to the recurrence relation $u_n = 2u_{n-1} - 2u_{n-2}$.

**b** Given that $u_0 = 1$ and $u_2 = 2$, find the particular solution to the recurrence relation.

**a** $r^2 - 2r + 2 = 0$

$\Rightarrow r = \dfrac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm \text{i} = \sqrt{2}e^{\pm\frac{\pi i}{4}}$

The auxiliary equation has distinct complex roots. These must be a conjugate pair, so you can write them in the form $x \pm \text{i}y$ or $re^{\pm \text{i}\theta}$.

← Core Pure Book 2, Chapter 1

Form 1: $u_n = A(1 + \text{i})^n + B(1 - \text{i})^n$

Form 2: $u_n = (\sqrt{2})^n\left(C\cos\dfrac{n\pi}{4} + D\sin\dfrac{n\pi}{4}\right)$

The form $u_n = r^n(A\cos n\theta + B\sin n\theta)$ only uses real numbers.

**b** Using form 1:

$u_0 = A(1 + i)^0 + B(1 - i)^0 = 1$

$\Rightarrow A + B = 1 \quad (1)$

$u_1 = A(1 + i)^1 + B(1 - i)^1 = 2$

$\Rightarrow A + B + (A - B)i = 2$

$\Rightarrow (A - B)i = 1$

$\Rightarrow A - B = -i \quad (2)$

Solving (1) and (2):

$A = \frac{1 - i}{2}$ and $B = \frac{1 + i}{2}$

So the particular solution is

$u_n = \left(\frac{1 - i}{2}\right)(1 + i)^n + \left(\frac{1 + i}{2}\right)(1 - i)^n$

Using form 2:

$u_0 = C(\sqrt{2})^0 \cos 0 + D(\sqrt{2})^0 \sin 0 = 1$

$\Rightarrow C = 1$

$u_1 = C(\sqrt{2})^1 \cos\frac{\pi}{4} + D(\sqrt{2})^1 \sin\frac{\pi}{4} = 2$

$\Rightarrow 1 \times \sqrt{2} \times \frac{\sqrt{2}}{2} + D \times \sqrt{2} \times \frac{\sqrt{2}}{2} = 2$

$\Rightarrow D = 1$

So the particular solution is

$u_n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right)$

**Watch out** The values of the arbitrary constants will be different depending on which form you use.

Use the initial conditions to find the values of the arbitrary constants $A$ and $B$. If you are using this form of the general solution, the arbitrary constants can be **complex**.

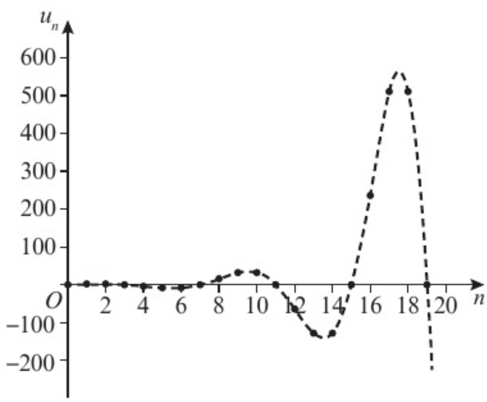
**Problem-solving**

You can simplify this to $u_n = (1 + i)^{n-1} + (1 - i)^{n-1}$ by writing, for example,

$\left(\frac{1 - i}{2}\right)(1 + i)^n = \frac{(1 - i)(1 + i)}{2}(1 + i)^{n-1} = (1 + i)^{n-1}$

With this form of the general solution, both arbitrary constants will be real numbers.

You can use the addition formula for sine to write the solution to the recurrence relation in Example 17 as $u_n = (\sqrt{2})^{n+1}\sin\frac{(n + 1)\pi}{4}$. This helps you to see that the sequence oscillates between positive and negative values, with the magnitude of the oscillations increasing as $n$ increases. The graph shows the sequence from $u_0$ to $u_{19}$. Note that the terms only exist for integer values of $n$ and that, in this case, $u_n$ is always an integer.




## Example 18

The Fibonacci sequence is defined recursively as

$$F_n = F_{n-1} + F_{n-2},\ n > 2, \text{ with } F_1 = 1 \text{ and } F_2 = 1$$

Find a closed form for $F_n$.

$r^2 - r - 1 = 0$

$\Rightarrow r = \frac{1 + \sqrt{5}}{2}$ or $r = \frac{1 - \sqrt{5}}{2}$

So the general solution is

$F_n = A\left(\frac{1 + \sqrt{5}}{2}\right)^n + B\left(\frac{1 - \sqrt{5}}{2}\right)^n$

Solve the auxiliary equation.

Using the initial conditions,

$F_1 = 1 \Rightarrow A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$A(1+\sqrt{5}) + B(1-\sqrt{5}) = 2 \qquad (1)$

$F_2 = 1 \Rightarrow A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1$

$A(3+\sqrt{5}) + B(3-\sqrt{5}) = 2 \qquad (2)$

Solving (1) and (2) simultaneously,

$A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$

The solution is

$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

**Watch out** You can solve these simultaneous equations quickly using your calculator. However, make sure you show enough working to demonstrate that you have used the initial conditions to generate two simultaneous equations.

You can solve **non-homogeneous** linear second-order recurrence relations by considering the complementary function (C.F.) and finding a suitable particular solution (P.S.).

- **To solve the recurrence relation $u_n = au_{n-1} + bu_{n-2} + \mathrm{g}(n)$,**
  - **Find the complementary function, which is the general solution to the associated homogeneous recurrence relation $u_n = au_{n-1} + bu_{n-2}$.**
  - **Choose an appropriate form for a particular solution then substitute into the original recurrence relation to find the values of any coefficients.**
  - **The general solution is $u_n$ = C.F. + P.S.**
  - **Use the initial conditions to find the values of the arbitrary constants.**

The form of the particular solution will depend on g($n$).

- **For the recurrence relation $u_n = au_{n-1} + bu_{n-2} + \mathrm{g}(n)$, with auxiliary equation with roots $\alpha$ and $\beta$, try the following forms for a particular solution:**

| Form of g($n$) | Form of particular solution |
|---|---|
| $p$ with $\alpha, \beta \neq 1$ | $\lambda$ |
| $pn + q$, with $\alpha, \beta \neq 1$ | $\lambda n + \mu$ |
| $kp^n$ with $p \neq \alpha, \beta$ | $\lambda p^n$ |
| $p$ with $\alpha = 1, \beta \neq 1$ | $\lambda n$ |
| $pn + q$ with $\alpha = 1, \beta \neq 1$ | $\lambda n^2 + \mu n$ |
| $p$ with $\alpha = \beta = 1$ | $\lambda n^2$ |
| $pn + q$ with $\alpha = \beta = 1$ | $\lambda n^3 + \mu n^2$ |
| $k\alpha^n$ with $\alpha \neq \beta$ | $\lambda n\alpha^n$ |
| $k\alpha^n$ with $\alpha = \beta$ | $\lambda n^2\alpha^n$ |

**Watch out** The particular solution cannot have any terms in common with the complementary function of the associated homogeneous recurrence relation. The last six lines of this table, shown shaded, are special cases to avoid this.
When $\alpha = 1$, multiply the expected form of the particular solution by $n$.
When $\alpha = 1$ and $\beta = 1$, multiply the expected form of the particular solution by $n^2$.

## Example 19

Solve the recurrence relation

$$a_{n+2} + 4a_{n+1} + 3a_n = 5(-2)^n,\ n > 0, \text{ with } a_0 = 2 \text{ and } a_1 = -1$$

Associated homogeneous recurrence relation:
$a_{n+2} + 4a_{n+1} + 3a_n = 0$
$r^2 + 4r + 3 = 0$
$(r + 1)(r + 3) = 0$
$\Rightarrow r = -1$ or $r = -3$
So the complementary function is
$a_n = A(-1)^n + B(-3)^n$
Try particular solution $a_n = \lambda(-2)^n$:
$a_{n+2} + 4a_{n+1} + 3a_n = 5(-2)^n$
$\lambda(-2)^{n+2} + 4\lambda(-2)^{n+1} + 3\lambda(-2)^n = 5(-2)^n$
$4\lambda - 8\lambda + 3\lambda = 5$
$\lambda = -5$
So a particular solution is $a_n = -5(-2)^n$, and the general solution to the recurrence relation is
$a_n = A(-1)^n + B(-3)^n - 5(-2)^n$
$a_0 = A(-1)^0 + B(-3)^0 - 5(-2)^0 \Rightarrow A + B - 5 = 2$
$A + B = 7$
$a_1 = A(-1)^1 + B(-3)^1 - 5(-2)^1 = -1$
$-A - 3B + 10 = -1$
$A + 3B = 11$
$\left.\begin{array}{r} A + B = 7 \\ A + 3B = 11 \end{array}\right\} \Rightarrow A = 5$ and $B = 2$
So the solution is $a_n = 5(-1)^n + 2(-3)^n - 5(-2)^n$

Find the general solution to the associated homogeneous recurrence relation.

Divide both sides of the equation by $(-2)^n$ and simplify.

General solution = C.F. + P.S.

The values of $A$ and $B$ can be found by using the initial conditions.

**Problem-solving**
Check your answer using $n = 0, 1, 2$ to make sure that it gives the same values as the recurrence relation.

## Example 20

Find the general solution to $s_n = 3s_{n-1} + 4s_{n-2} + 4^n$.

Associated homogeneous recurrence relation:
$s_n - 3s_{n-1} - 4s_{n-2} = 0$
$r^2 - 3r - 4 = 0$
$(r + 1)(r - 4) = 0$
$\Rightarrow r = -1$ or $r = 4$
So the complementary function is
$s_n = A(-1)^n + B(4^n)$
Try particular solution $s_n = \lambda n(4^n)$:
$s_n = 3s_{n-1} + 4s_{n-2} + 4^n$
$\lambda n(4^n) = 3\lambda(n - 1)(4^{n-1}) + 4\lambda(n - 2)(4^{n-2}) + 4^n$
$\lambda n(4^n) = \frac{3}{4}\lambda(n - 1)(4^n) + \frac{1}{4}\lambda(n - 2)(4^n) + 4^n$
$\lambda n = \frac{3}{4}\lambda(n - 1) + \frac{1}{4}\lambda(n - 2) + 1$
$\lambda n = \frac{3}{4}\lambda n - \frac{3}{4}\lambda + \frac{1}{4}\lambda n - \frac{1}{2}\lambda + 1$
So $-\frac{5}{4}\lambda + 1 = 0 \Rightarrow \lambda = \frac{4}{5}$
So a particular solution is $\frac{4}{5}n(4^n)$, and the general solution is
$s_n = A(-1)^n + B(4^n) + \frac{4}{5}n(4^n)$

Write down the auxiliary equation and solve it.

**Watch out** You cannot use a particular solution of the form $\lambda(4^n)$ because the complementary function already features a $4^n$ term. Look for a particular solution of the form $\lambda n(4^n)$ instead.

Substitute the particular solution into the full recurrence relation and solve to find $\lambda$.

This question only asks for the general solution, so leave your answer in this form with two arbitrary constants.

## Exercise 7C

A

1 Consider the recurrence relation $u_n = 5u_{n-1} + 6u_{n-2}$. Verify that each of the following solutions satisfies this recurrence relation.

**a** $u_n = (-1)^n$ **b** $u_n = 6^n$

**c** $u_n = A(-1)^n + B(6^n)$, where $A$ and $B$ are arbitrary constants.

2 Consider the recurrence relation $u_n - 6u_{n-1} + 9u_{n-2} = 0$. Verify that each of the following solutions satisfies this recurrence relation.

**a** $u_n = 5(3^n)$ **b** $u_n = -n3^n$ **c** $u_n = 5(3^n) - n3^n$

3 Consider the recurrence relation $u_{n+2} + u_n = 0$. Verify that each of the following solutions satisfies this recurrence relation.

**a** $u_n = \cos\left(n\frac{\pi}{2}\right)$ **b** $u_n = \sin\left(n\frac{\pi}{2}\right)$

**c** $u_n = A\cos\left(n\frac{\pi}{2}\right) + B\sin\left(n\frac{\pi}{2}\right)$, where $A$ and $B$ are arbitrary constants.

P 4 $u_n = \mathrm{F}(n)$ and $u_n = \mathrm{G}(n)$ are particular solutions to the linear homogeneous recurrence relation

$$u_n = au_{n-1} + bu_{n-2}$$

Show that $u_n = c\mathrm{F}(n) + d\mathrm{G}(n)$ is also a solution, where $c$ and $d$ are arbitrary constants.

5 Find the general solution to each of the following recurrence relations.

**a** $a_n = 2a_{n-1} - a_{n-2}$ **b** $u_n - 3u_{n-1} + 2u_{n-2} = 0$

**c** $x_n = 6x_{n-1} - 9x_{n-2}$ **d** $t_n = 4t_{n-1} - 5t_{n-2}$

**Hint** Your general solutions will each contain two arbitrary constants.

P 6 The recurrence relation $u_{n+2} + au_{n+1} + bu_n = 0$, where $a$ and $b$ are real constants, has general solution $u_n = D + E(7^n)$, where $D$ and $E$ are arbitrary constants.
Find the values of $a$ and $b$.

7 Solve each of the following recurrence relations.

**a** $a_n = 5a_{n-1} - 6a_{n-2}$, with $a_0 = 2$ and $a_1 = 5$

**b** $u_n = 6u_{n-1} - 9u_{n-2}$, $n \geqslant 3$, with $u_1 = 2$ and $u_2 = 5$

**c** $s_n = 7s_{n-1} - 10s_{n-2}$, $n \geqslant 2$, with $s_0 = 4$ and $s_1 = 17$

**d** $u_n = 2u_{n-1} - 5u_{n-2}$, with $u_0 = 1$ and $u_1 = 5$

E/P 8 A sequence satisfies the recurrence relation $u_n = 5u_{n-1} - 4u_{n-2}$, with $u_0 = 20$ and $u_1 = 19$.

**a** Solve the recurrence relation to find a closed form for $u_n$. **(5 marks)**

**b** Show that the sequence is decreasing, and that $u_n < 0$ for all $n \geqslant 3$. **(3 marks)**

E/P 9 **a** Find a closed form for the sequence defined by the recurrence relation
$u_n = \sqrt{2}u_{n-1} - u_{n-2}$, with $u_0 = u_1 = 1$ **(5 marks)**

**b** Hence show that the sequence is periodic and state its period. **(3 marks)**

E 10 The $n$th Lucas number $L_n$, is defined by $L_n = L_{n-1} + L_{n-2}$, $n \geqslant 3$, with $L_1 = 1$, $L_2 = 3$.

**a** List the first 7 terms of the sequence. **(1 mark)**

**b** Show that a closed form for the $n$th Lucas number is $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$ **(5 marks)**

A **11** Find the general solution to each of the following recurrence relations.

**a** $x_n = 5x_{n-1} - 6x_{n-2} + 1$ **b** $u_n - u_{n-1} - 2u_{n-2} = 2n$

**c** $a_{n+2} + 4a_{n+1} + 3a_n = 5(-2)^n$ **d** $a_{n+2} + 4a_{n+1} + 3a_n = 12(-3)^n$

**e** $a_{n+2} - 6a_{n+1} + 9a_n = 3^n$ **f** $u_n = 7u_{n-1} - 10u_{n-2} + 6 + 8n$

**Problem-solving**

Each of these recurrence relations will require a particular solution. Look at the table on page 232 to determine the correct form for the particular solution.

**12** Solve each of the following recurrence relations.

**a** $u_n = 2u_{n-1} + 3u_{n-2} + 1, n \geqslant 3$, with $u_1 = 3$ and $u_2 = 7$

**b** $a_{n+1} - 3a_n + 2a_{n-1} = 6(-1)^n$, with $a_0 = a_1 = 12$

**c** $u_n = 3u_{n-1} + 10u_{n-2} + 7 \times 5^n, n \geqslant 2$, with $u_0 = 4$ and $u_1 = 3$

**d** $x_n = 10x_{n-1} - 25x_{n-2} + 8 \times 5^n, n \geqslant 2$, with $x_0 = 6$ and $x_1 = 10$

E/P **13** Consider the recurrence relation $b_{n+2} + 4b_{n+1} + 4b_n = 7$.

**a** Find a constant $k$ such that $b_n = k$ is a particular solution to this recurrence relation. **(2 marks)**

**b** Hence or otherwise, solve the recurrence relation given that $b_0 = 1$ and $b_1 = 2$. **(5 marks)**

E/P **14** **a** Find the general solution to the recurrence relation $u_n = 7u_{n-1} - 6u_{n-2} + 75$. **(4 marks)**

**b** Given that $u_0 = u_1 = 2$, find the particular solution. **(3 marks)**

E/P **15** Consider the recurrence relation $u_{n+2} - 6u_{n+1} + 9u_n = 7(3^n)$.

**a** Find a value of $k$ such that $u_n = kn^2(3^n)$ is a particular solution to this recurrence relation. **(2 marks)**

**b** Find the general solution to $u_{n+2} - 6u_{n+1} + 9u_n = 0$. **(3 marks)**

**c** Hence, find the solution to $u_{n+2} - 6u_{n+1} + 9u_n = 7(3^n)$ given that $u_0 = 1$ and $u_1 = 4$. **(3 marks)**

E/P **16** A sequence of numbers satisfies the recurrence relation $u_n = u_{n-1} - u_{n-2}, n \geqslant 2$.

**a** Given that $u_0 = 0$ and $u_1 = 3$, show the solution to this recurrence relation can be written in the form $u_n = p\sin qn$, where $p$ and $q$ are exact real constants to be determined. **(6 marks)**

**b** Hence explain why the sequence $u_n$ is periodic, and state its period. **(2 marks)**

E/P **17** A monkey sits at a typewriter and types strings of random letters. Unfortunately, the typewriter is broken, so the only keys that work are the letters A, B and C.

**a** Find the number of different strings of length 3 which do not contain consecutive letter As. **(2 marks)**

The number of different strings of length $n$ which do not contain consecutive letter As is given by $s_n$.

**b** Find a recurrence relation for $s_n$ in terms of $s_{n-1}$ and $s_{n-2}$. **(3 marks)**

**c** **i** Solve your recurrence relation.

**ii** Find the number of strings of length 20 which do not contain consecutive letter As. **(3 marks)**

**Challenge**

**1** Solve the recurrence relation $u_n = \sqrt{\frac{u_{n-2}}{u_{n-1}}}$, $u_0 = 8$, with $u_1 = \frac{1}{2\sqrt{2}}$

**2** The sequence $u_n$ satisfies the recurrence relation $u_n = au_{n-1} + bu_{n-2}$, with $u_0 = 0$ and $u_1 = k$, where $a$, $b$ and $k$ are real constants, and $k \neq 0$. Find values of $a$ and $b$ such that the sequence is periodic with period 12, and state the maximum and minimum values in the sequence in terms of $k$.

**Hint** Take logs of both sides, and then use a suitable substitution to form a linear recurrence relation.

## Mixed exercise 7

(E) **1** Solve the recurrence relation $u_n - 2u_{n-1} + 1 = 0$,with $u_0 = 4$. **(3 marks)**

(E/P) **2** **a** Solve the recurrence relation $u_n = u_{n-1} - n$, with $u_0 = 2000$. **(3 marks)**

**b** Hence, or otherwise, find the first negative term of the sequence. **(2 marks)**

(E/P) **3** **a** Solve the recurrence relation $u_n = 3u_{n-1} + 5$, with $u_0 = 0$. **(3 marks)**

**b** Find $u_{10}$. **(1 mark)**

**c** Find the first term of the sequence to exceed 10 million. **(2 marks)**

(E/P) **4** At the end of each year, a sustainable lumber company harvests 20% of its trees. To replace this stock they plant 1000 new trees. At the beginning of the first year, the company has 12 000 trees.

Let $T_n$ represent the number of trees remaining at the end of the $n$th year.

**a** Explain why the number of trees owned by the company can be modelled by the recurrence relation $T_n = 0.8T_{n-1} + 1000$, with $T_0 = 12000$. **(3 marks)**

**b** Solve this recurrence relation to find a closed form for $T_n$. **(3 marks)**

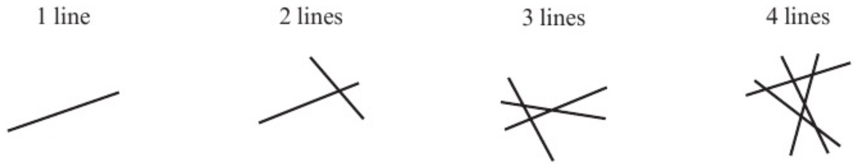
**c** In the long run, how many trees can the lumber company expect to have at the end of each year? **(1 mark)**

(E/P) **5** The Smiths buy a new house in March 2018, costing £200 000. They have a deposit of £25 000. At the end of each month, interest of 0.25% is added to the balance, and the Smiths' monthly payment of £1200 is deducted from the balance.

**a** Write a recurrence relation showing the balance in pounds, $b_n$, at the end of the $n$th month. **(3 marks)**

**b** By solving your recurrence relation, determine the year in which the Smiths will pay off their mortgage. **(5 marks)**

(E/P) **6** The diagrams show intersecting lines drawn on a two-dimensional plane.




Assuming that all the lines are non-parallel, and that no three lines intersect at a common point, the number of points of intersection when $n$ lines are drawn, $P_n$, is given by

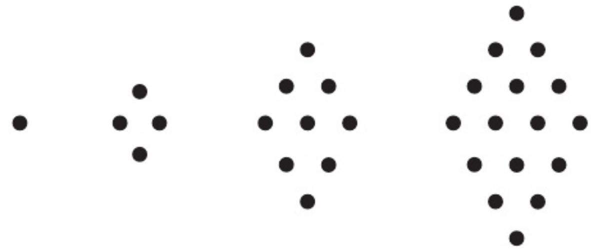
$P_1 = 0, \quad P_2 = 1, \quad P_3 = 3$

**a** Use the diagram above to write down the value of $P_4$. **(1 mark)**

**b** By forming and solving a suitable recurrence relation, show that $P_n = \frac{1}{2}n(n-1)$. **(4 marks)**

**c** Hence find the number of intersections formed when 100 such lines are drawn on the plane. **(1 mark)**

E/P **7** A sequence of patterns is formed by drawing dots in the shape of a rhombus, as shown in the diagram. The number of dots needed to draw the $n$th shape is represented by $t_n$.



**a** Write down the values of $t_5$, $t_6$, and $t_7$. **(1 mark)**

**b** Find in term of $t_{n-1}$, the recurrence relation for $t_n$. **(2 marks)**

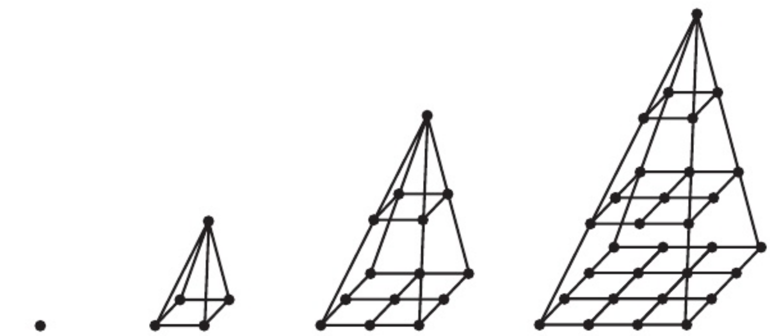
**c** Solve your recurrence relation for $t_n$, and hence determine the number of dots in the 100th pattern. **(3 marks)**

E/P **8** **a** Calculate $\begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}$, giving your answer as a $2 \times 2$ matrix with elements given in terms of $p$ and $q$. **(3 marks)**

**b** Given that $\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & a_n \\ 0 & b_n \end{pmatrix}$, write down recurrence relations for $a_n$ in terms of $a_{n-1}$ and $b_n$ in terms of $b_{n-1}$. **(2 marks)**

**c** Solve your recurrence relations, and hence find $\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}^n$, giving your answer as a $2 \times 2$ matrix with elements given in terms of $n$. **(4 marks)**

E/P **9** Square pyramidal numbers $S_n$ are positive integers that can be represented by square pyramidal shapes. The first four square pyramidal numbers are 1, 5, 14, and 30, as shown in the diagram below.




**a** Write down $S_5$, $S_6$, and $S_7$. **(1 marks)**

**b** Find a recurrence relation for $S_n$ in terms of $S_{n-1}$. **(2 marks)**

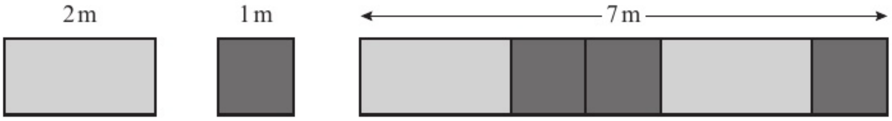
**c** Solve your recurrence relation to find a closed form for $S_n$. **(3 marks)**

E/P **10** In a drug trial, a bacterial population is modelled as increasing at a rate of 20% each hour. A proposed antibacterial agent is introduced, and kills bacteria at a rate of $k(2^n)$ bacteria per hour, where $k$ is a measure of the concentration of the agent.
At the beginning of the trial there are 100 bacteria present, and after $n$ hours there are $u_n$ bacteria present.

**a** Form a recurrence relation for $u_n$ in terms of $u_{n-1}$, stating the initial condition. **(2 marks)**

**b** Show that $u_n = \left(100 + \frac{5k}{2}\right)(1.2^n) - \frac{5k}{2}(2^n)$ **(5 marks)**

A E/P **11** Flagstones come in two different sizes. Large flagstones have a length of 2 m, and small flagstones have a length of 1 m. The diagram shows a large and small flagstone, and a path of length 7 m made from a combination of these flagstones.




Let $f_n$ represent the number of ways in which a path of length $n$ m can be made from a combination of large and small flagstones.

**a** Draw the three possible paths of length 3 m. **(1 mark)**

**b** Explain why $f_n$ satisfies the recurrence relation

$f_n = f_{n-1} + f_{n-2}$, with $f_1 = 1$ and $f_2 = 2$ **(3 marks)**

A path of length 200 m is to be made.

**c** Show that the number of ways in which this path could be constructed is given by

$$\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{201} - \left(\frac{1-\sqrt{5}}{2}\right)^{201}\right)$$ **(5 marks)**

E/P **12** A **ternary string** is a sequence of digits, where each digit can be either 0, 1 or 2. There are 8 different ternary strings of length 2 which **do not** contain consecutive 0s.

01, 10, 02, 20, 11, 12, 21, 22

Let $t_n$ represent the number of ternary strings of length $n$ with no consecutive 0s.

**a** Find $t_2$ and $t_3$. **(1 mark)**

**b** Explain why $t_n$ satisfies the recurrence relation $t_n = 2t_{n-1} + 2t_{n-2}$. **(3 marks)**

**c** Find $t_6$. **(1 mark)**

**d** Find:

**i** a closed form for $t_n$ in terms of $n$

**ii** the number of different ternary strings length 15 which do not contain consecutive 0s. **(5 marks)**

E **13** **a** Find the general solution to the recurrence relation

$u_{n+2} = u_{n+1} + 2u_n$, $n \geqslant 1$ **(3 marks)**

**b** Given that $u_1 = 1$ and $u_2 = 2$, find the particular solution to the recurrence relation. **(3 marks)**

E/P **14** **a** Find the general solution to the recurrence relation

$x_{n+2} = 7x_{n+1} - 10x_n + 3$, $n \geqslant 1$ **(4 marks)**

**b** Given that $x_1 = 1$ and $x_2 = 2$, find the particular solution to the recurrence relation. **(3 marks)**

E/P **15** Solve the recurrence relation $a_n = 2a_{n-1} + 15a_{n-2} + 2^n$, with $a_1 = 2$ and $a_2 = 4$. **(8 marks)**

E/P **16** **a** Find a closed form for the sequence defined by the recurrence relation

$u_n = \sqrt{2}u_{n-1} - u_{n-2}$, with $u_0 = u_1 = 1$ **(5 marks)**

**b** Hence show that the sequence is periodic and state its period. **(3 marks)**

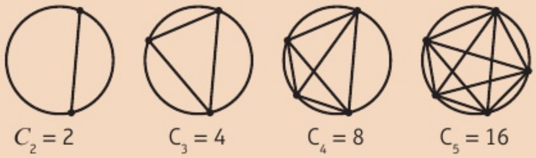
A E/P **17** Messages are transmitted over a network using two types of signal packet. Type $A$ signal packets require 1 microsecond to transmit, and type $B$ signal packets require 2 microseconds to transmit. The packets are transmitted consecutively with no gaps between them.
The number of different messages consisting of sequences of these two types of signal packet that can be sent in $n$ microseconds is denoted by $S_n$.

**a** Write a recurrence relation for $S_{n+2}$ in terms of $S_{n+1}$ and $S_n$. State the initial conditions for your recurrence relation. **(3 marks)**

**b** Solve your recurrence relation to find an expression for $S_n$ in terms of $n$. **(5 marks)**

## Challenge

**1** The circles in the diagram have been subdivided into regions by drawing all possible chords between points drawn on the circumference of the circle. Let $C_n$ denote the maximum number of regions that are formed in this way when $n$ points are drawn on the circumference of a circle.




**a** Find $C_6$.

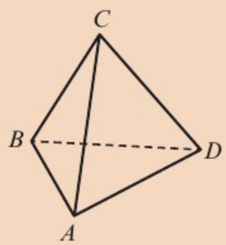
**b** Explain why it is always possible to choose a new point on the circumference of the circle such that all the new chords drawn from that point do not intersect the existing chords at any existing points of intersection.

**c** Find, in terms of $C_{n-1}$ and $n$, a recurrence relation for $C_n$.

**d** Solve your recurrence relation, and hence determine the maximum number of regions created by 100 points.

**2** The diagram shows a tetrahedron $ABCD$.
A spider walks along the edges of the tetrahedron, starting and ending at vertex $A$. A walk of length $n$ traverses exactly $n$ edges, so that there are three possible walks of length 2:

$A \to B \to A$
$A \to C \to A$
$A \to D \to A$




**a** Explain why there are no possible walks of length 1.

**b** Find the number of possible walks of length 3.

**c** By formulating and solving a suitable recurrence relation find a closed form for the total number of possible walks of length $n$.

## Summary of key points

1 A recurrence relation is an equation that defines a sequence based on a rule that gives the each term as a function of the previous term(s).

2 The **order** of a recurrence relation is the difference between the highest and lowest subscripts in the relation.

3 A sequence $u_n$ is called a **solution** to a recurrence relation if its terms satisfy the recurrence relation. It is also called the **closed form** of the sequence.

4 A **first-order** recurrence relation is one in which $u_n$ can be given as a function of $n$ and $u_{n-1}$ only.

5 A **first-order linear** recurrence relation can be written in the form $\boldsymbol{u_n = au_{n-1} + \mathrm{g}(n)}$.

- If $g(n) = 0$, then the equation is homogeneous.
- The solution to the first-order homogeneous linear recurrence relation $u_n = au_{n-1}$ is given by $u_n = u_0a^n$ or $u_n = u_1a^{n-1}$.
- The solution to the first-order non-homogeneous linear recurrence relation $u_n = u_{n-1} + \mathrm{g}(n)$ is given by $u_n = u_0 + \sum_{r=1}^{n} \mathrm{g}(r)$.
- When solving a recurrence relation of the form $u_n = au_{n-1} + \mathrm{g}(n)$, the form of the particular solution will depend on $\mathrm{g}(n)$:

| Form of $\mathrm{g}(n)$ | Form of particular solution |
|---|---|
| $p$ with $a \neq 1$ | $\lambda$ |
| $pn + q$ with $a \neq 1$ | $\lambda n + \mu$ |
| $kp^n$ with $p \neq a$ | $\lambda p^n$ |
| $ka^n$ | $\lambda na^n$ |

6 To solve the recurrence relation $\boldsymbol{u_n = au_{n-1} + \mathrm{g}(n)}$,

- Find the complementary function, which is the general solution to the associated homogeneous recurrence relation $u_n = au_{n-1}$.
- Choose an appropriate form for a particular solution then substitute into the original recurrence relation to find the values of any coefficients.
- The general solution is $u_n = \text{C.F.} + \text{P.S.} = ca^n + \text{P.S.}$
- Use the initial condition to find the value of the arbitrary constant.

7 If $u_n = \mathrm{F}(n)$ and $u_n = \mathrm{G}(n)$ are particular solutions to a linear recurrence relation, then $u_n = a\mathrm{F}(n) + b\mathrm{G}(n)$, where $a$ and $b$ are constants, is also a solution.

A

8 A **second-order linear** recurrence relation can be written in the form $\boldsymbol{u_n = au_{n-1} + bu_{n-2} + \mathrm{g}(n)}$, where $a$ and $b$ are real constants.

- If $\mathrm{g}(n) = 0$, then the equation is homogeneous.

A

**9** You can find a general solution to a **second-order homogeneous** linear recurrence relation, $u_n = au_{n-1} + bu_{n-2}$, by considering the auxiliary equation, $r^2 - ar - b = 0$.
You need to consider three different cases:

**Case 1: Distinct real roots**
If the auxiliary equation has distinct real roots $\alpha$ and $\beta$, then the general solution will have the form $u_n = A\alpha^n + B\beta^n$ where $A$ and $B$ are arbitrary constants.

**Case 2: Repeated root**
If the auxiliary equation has a repeated real root $\alpha$, then the general solution will have the form $u_n = (A + Bn)\alpha^n$ where $A$ and $B$ are arbitrary constants.

**Case 3: Complex roots**
If the auxiliary equation has two complex roots $\alpha = r\mathrm{e}^{\mathrm{i}\theta}$ and $\beta = r\mathrm{e}^{-\mathrm{i}\theta}$, then the general solution will have the form $u_n = r^n(A\cos n\theta + B\sin n\theta)$, or $u_n = A\alpha^n + B\beta^n$, where $A$ and $B$ are arbitrary constants.

**10** To solve the recurrence relation $\boldsymbol{u_n = au_{n-1} + bu_{n-2} + \mathrm{g}(n)}$,

- Find the complementary function, which is the general solution to the associated homogeneous recurrence relation $u_n = au_{n-1} + bu_{n-2}$.
- Choose an appropriate form for a particular solution then substitute into the original recurrence relation to find the values of any coefficients.
- The general solution is $u_n$ = C.F. + P.S.
- Use the initial conditions to find the values of the arbitrary constants.

**11** For the recurrence relation $u_n = au_{n-1} + bu_{n-2} + \mathrm{g}(n)$, with auxiliary equation with roots $\alpha$ and $\beta$, try the following forms for a particular solution:

| Form of g($n$) | Form of particular solution |
|---|---|
| $p$ with $\alpha, \beta \neq 1$ | $\lambda$ |
| $pn + q$, with $\alpha, \beta \neq 1$ | $\lambda n + \mu$ |
| $kp^n$ with $p \neq \alpha, \beta$ | $\lambda p^n$ |
| $p$ with $\alpha = 1, \beta \neq 1$ | $\lambda n$ |
| $pn + q$ with $\alpha = 1, \beta \neq 1$ | $\lambda n^2 + \mu n$ |
| $p$ with $\alpha = \beta = 1$ | $\lambda n^2$ |
| $pn + q$ with $\alpha = \beta = 1$ | $\lambda n^3 + \mu n^2$ |
| $k\alpha^n$ with $\alpha \neq \beta$ | $\lambda n\alpha^n$ |
| $k\alpha^n$ with $\alpha = \beta$ | $\lambda n^2\alpha^n$ |

# Decision analysis

# 8

**Objectives**

After completing this chapter you should be able to:

- Use, construct and interpret decision trees → pages 243–252
- Calculate expected monetary values (EMVs) → pages 243–252
- Use utility to compare different courses of action → pages 252–259


Making decisions effectively requires an understanding of the likely consequences associated with each possible choice. Decision analysis is used frequently in business and finance to make investment decisions.

→ Exercise 8B, Q2

**Prior knowledge check**

**1** Calculate the expected score when an ordinary six-sided dice is rolled once. ← GCSE Mathematics

**2** The table shows how the number of bricks that a bricklayer can lay in one day is dependent on the weather.

| **Weather** | Good | Fair | Poor |
|---|---|---|---|
| **Number of bricks** | 800 | 600 | 400 |

On a particular day, the probability that the weather will be good is 0.7, the probability that it will be fair is 0.2 and the probability that it will be poor is 0.1.

Calculate the expected number of bricks that will be laid in the day. ← GCSE Mathematics

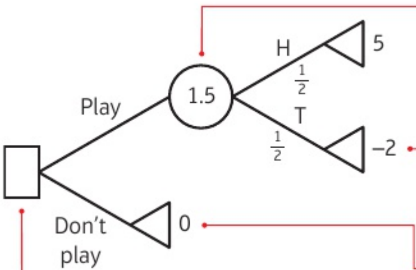
## 8.1 Decision trees

A In probability, you use tree diagrams to show all the possible outcomes of two or more events, and their probabilities. You can extend the concept of a tree diagram to include **decisions**. This can help you determine the expected pay-off for different decisions in a game or real-life situation.

- **In a decision tree:**
  - **boxes represent decision nodes**
  - **triangles represent end (or pay-off) nodes**
  - **circles represent chance nodes**
- **You annotate the decision tree to show the following information:**
  - **For an event, the probabilities of each outcome are given on the branches.**
  - **The pay-off for each end node is written next to that node.**
  - **The expected value of each event is written inside the chance node for that event. This value is called the expected monetary value (EMV).**

**Watch out** Probability tree diagrams also show probabilities on branches but are otherwise very different from decision trees. You should never need to draw a probability tree diagram in your D2 exam.

Here is an example of a decision tree for a simple game in which a player tosses a coin and wins £5 for heads, or loses £2 for tails.




Chance nodes appear before an **event**. The number in the chance node represents the expected value of this event. In this case, the expected value of playing the game is $5 \times \frac{1}{2} - 2 \times \frac{1}{2} = 1.5$. This is the **expected monetary value (EMV)** of the decision to play the game.

If the coin lands on tails the player **loses money**, so the pay-off is given as a negative number.

If the player does not play the game then the decision tree **ends**. In this case, no money is won or lost, so the pay-off is 0.

A decision node is used immediately before a **decision**. In this case, the player can choose to play the game or not play the game.

Example 1 illustrates a simple decision tree involving no chance nodes.

### Example 1

Carol plans to travel from Cannock to Derby by car. One option is to take the A460 to Uttoxeter and then take the A50. This is a distance of 38 miles and will take around 1 hour 20 minutes. A second option is to set off on the A5 to the M6 Toll road then continue on the A38. This is a distance of 36 miles and will take around 55 minutes. A final option is to combine the routes by taking the A460 to Rugeley, then the A515, then take the A38 at Alrewas. This is a distance of 34 miles and will take around 1 hour and 5 minutes.

**a** Use a decision tree to choose the best option for Carol based purely on:

**i** the distance to travel **ii** the time needed to complete the journey.

**A** The cost of driving is estimated at 40p per mile

There is an additional cost of £10 for using the M6 Toll road.

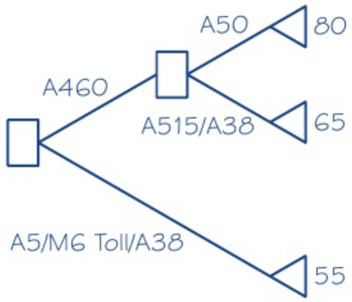
**b** Comment on the best option overall.

**a** **i** Using distance


> There are two decisions to be made in this problem, and each one has two possible branches. The pay-offs at the end nodes are the total distance travelled assuming those decisions are made.

Carol's best option, using distance as the criterion, is to take the A460 to the A515 and then take the A38.

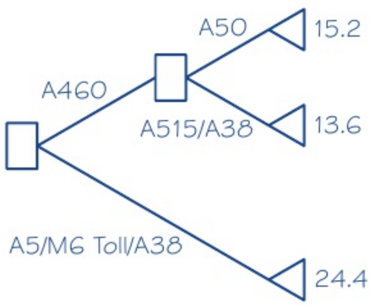
**ii** Using time




> The nature of the problem determines the pay-offs. If Carol wants to optimise the time needed, then the pay-off should be given in terms of time.

Carol's best route, using time as the criterion, is to take the A5 to the M6 Toll road and then the A38.

**b** Using cost




> You might need to carry out calculations to determine the pay-off. In this case, the pay-off (cost) of the decision to take the A5/M6 Toll/A38 route is $0.4 \times 36 + 10 = 24.4$, or £24.40.

Carol's best route, using cost as the criterion, is to take the A460 to the A515 and then take the A38.

Carol's final decision will depend on how important it is to save around 10 minutes of journey time at the expense of almost £11 in cost.

> Write a conclusion based on all the available analysis.

A You will usually need to select an optimal strategy based on your decision tree. To do this you compare the EMVs of different decisions, and choose the options that optimise the EMV.

- **An optimal strategy based on the EMV criterion is a set of decisions chosen so as to optimise the EMV.**

One common application of decision trees involving chance is to make a decision about whether or not to play a game, and to select an optimal strategy for that game.

## Example 2

James is trying to decide whether or not to play a game using two fair tetrahedral dice. The sides of the dice are numbered from 1 to 4. If he scores a total of 6 or more then he wins £4; otherwise he loses £2.

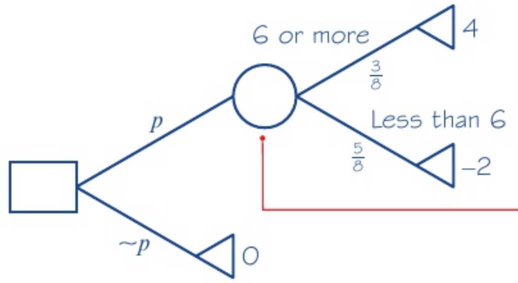
**a** Draw a decision tree to model James's possible decisions and outcomes.

**b** Determine whether James should play the game and state his EMV.

**a** The winning scores are (4, 2), (2, 4), (3, 3), (4, 3), (3, 4), (4, 4).

P(6 or more) = $\frac{6}{16} = \frac{3}{8}$, P(less than 6) = $\frac{5}{8}$

Start by working out the probabilities for the two outcomes of the event. There are 4 × 4 = 16 possible outcomes, and 6 of these produce a total of 6 or more.




**Notation** You can use $p$ to represent the decision to play and $\sim p$ to represent the decision not to play.

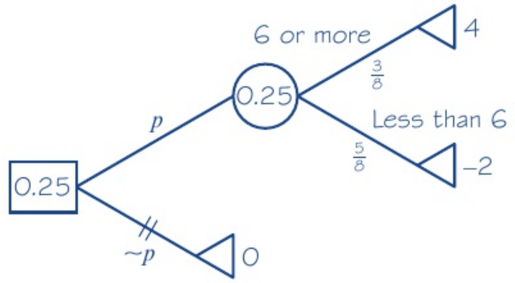
Draw a chance node before the event. Write the outcomes and their probabilities on the branches and write the pay-off for each outcome at the end node. A loss of £2 is written as –2.

If James plays the game his expected winnings are

$\frac{3}{8} \times 4 + \frac{5}{8} \times (-2) = 0.25$

On average, James can expect to win £0.25 per game that he plays.

You need to add an EMV to the chance node. This will be the expected value of the game.




**Notation** You can show the **optimal strategy** on a decision tree by crossing off non-optimal decisions. For each decision node, draw a double line through the decision which produces the lowest pay-off (or EMV), and write the value of the optimal pay-off (or EMV) inside the decision node. In this case, the decision to play is optimal, and the value of this decision is 0.25.

**b** The decision to play generates a positive EMV of £0.25, so James should choose to play the game.

## Example 3

A

Jess pays £2 to play a game. She rolls two dice and if they both show the same score, she wins £5; otherwise she loses.

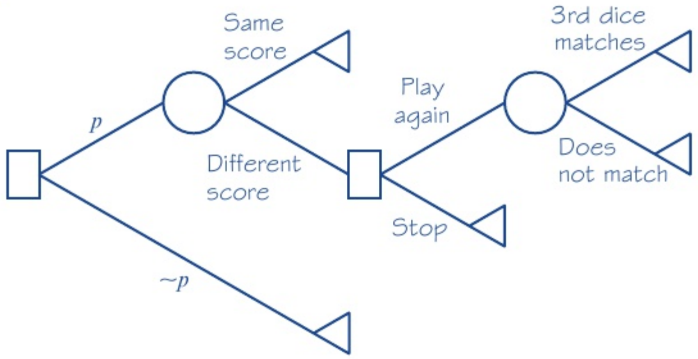
If she loses, she can pay a further £1 to roll a third dice. If the score on the third dice matches the score on either of the first two dice, then she wins £6; otherwise she loses.

**a** Draw a decision tree to model the decisions and possible outcomes.

**b** Find the best strategy for Jess as indicated by the decision tree.

**Online** Explore EMV using Geogebra.

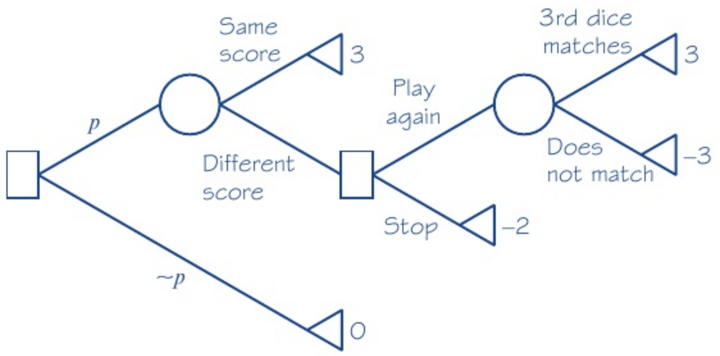
**a** The structure of the decision tree shows the decisions and possible outcomes.




**Problem-solving**

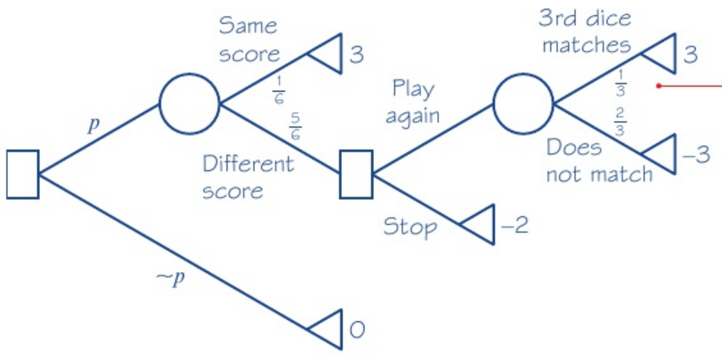
After the first event, there is a second decision to be made. Add a second decision node at the end of the losing outcome.

Now add the pay-offs to the decision tree.



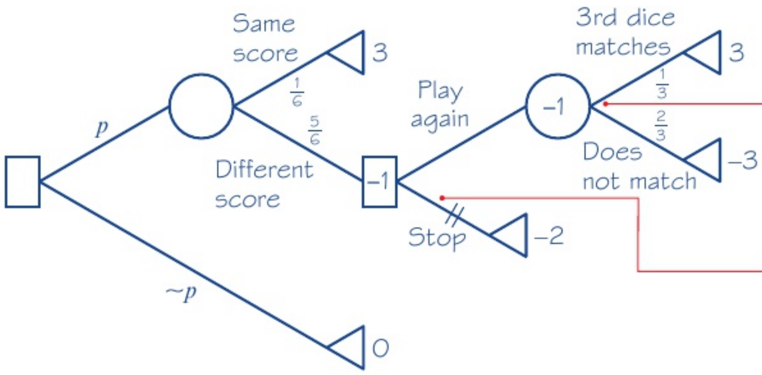

The pay-off for each outcome is given by: profit = winnings − amount paid.

Now add the probabilities to the branches to the right of each chance node.




The probability of two ordinary dice landing on the same value is $\frac{6}{36} = \frac{1}{6}$. The third dice is only rolled if the first two are showing different values, so the probability that it matches either of these values is $\frac{2}{6} = \frac{1}{3}$

Now add EMVs to the chance nodes.

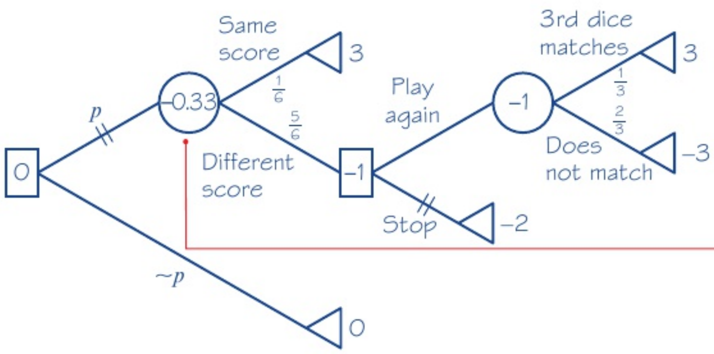



The EMVs are calculated working from right to left.

$\text{EMV} = \frac{1}{3} \times 3 + \left(\frac{2}{3} \times -3\right) = -1$

The EMV for continuing the game (−1) is better than the alternative (−2) of not continuing. This is shown by striking two lines through the 'Stop' branch and writing −1 in the decision node.

Continue to work from right to left to complete the decision tree.




**Watch out** Do not cross out any outcomes leading **from** chance nodes. These are chance events which you cannot control, so do not form part of your strategy.

$$\text{EMV} = \frac{1}{6} \times 3 + \left(\frac{5}{6} \times -1\right)$$
$$= -0.33$$

**b** The best strategy is not to play the game, since the overall EMV of playing is less than £0 for not playing. However, if Jess decides to play, and she loses at the first attempt, then she should pay £1 and roll the third dice.

## Example 4

Jack has time for one revision session before he takes an exam in Mechanics and Statistics.

He will devote this time either to Mechanics or Statistics. These are worth 50 marks each.

Jack estimates that if the Mechanics questions go in his favour then he would expect to achieve 45 marks, but if they don't then this reduces to 30 marks. Similarly, if the Statistics questions go in his favour then he would expect to achieve 35 marks, but if they don't then this reduces to 25 marks.

If Jack devotes his revision time to Mechanics then he has a 70% chance of finding the Mechanics questions favourable and a 20% chance of finding the Statistics questions favourable.

If Jack devotes his revision time to Statistics then he has a 60% chance of finding the Statistics questions favourable and a 60% chance of finding the Mechanics questions favourable.

Use a decision tree to determine how Jack should devote his revision time.

**A**

Use the following notation.

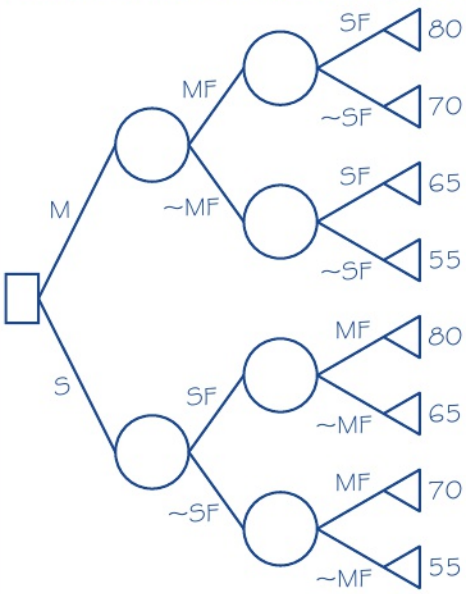
M: Revise Mechanics

MF: Mechanics questions are favourable

~MF: Mechanics questions are not favourable.

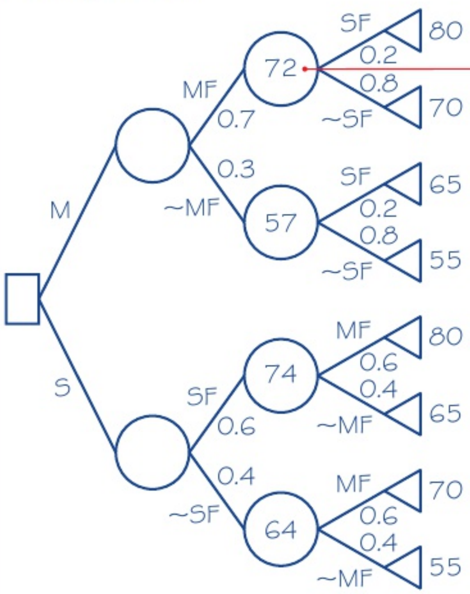
Use corresponding notation for Statistics.

The decision tree structure, with pay-offs, is



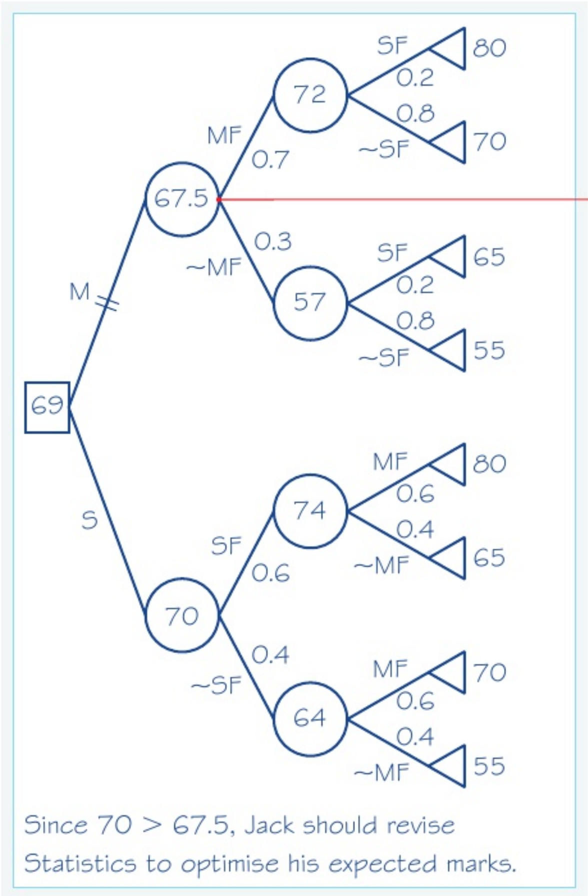

The pay-offs are Jack's expected total mark. If the Mechanics questions and the Statistics questions are both favourable then Jack's expected score is 45 + 35 = 80.

Now include the probabilities at the chance nodes. Calculate the expected values working from right to left.




Expected value = $0.2 \times 80 + 0.8 \times 70 = 72$

A

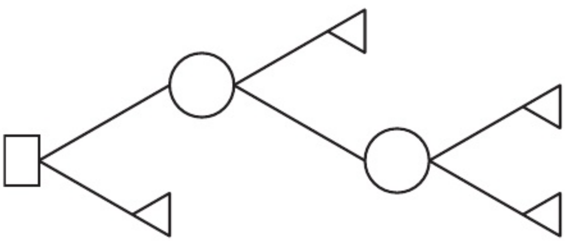



Expected value = 0.7 × 72 + 0.3 × 57 = 67.5

## Exercise 8A

**1** Claire considers playing a game for £2. To win, she must spin three coins that land the same way up. If she wins on her first go then she wins £6; otherwise she can have a second go but can then only win £4. She does not need to pay £2 again for her second go.

**a** Copy and complete the following decision tree to represent this situation.



**b** Describe Claire's optimal strategy.

**Hint** Your completed decision tree should have:

- labels for every decision and outcome
- probabilities for every outcome of a chance event
- pay-offs at every end node
- EMVs at every chance node
- EMVs at every decision node
- double lines through non-optimal decisions

A **2** Stephen is self-employed and is considering whether he should have the flu vaccine.

E Considering his age-group and lifestyle, Stephen estimates that he has a 30% chance of catching the flu if he is not vaccinated. His risk is reduced by a half if he is vaccinated.

The cost of the vaccination is £10. If he catches the flu, Stephen estimates that this will cost him £1000 in lost earnings.

**a** Draw a decision tree to model the situation. **(4 marks)**

**b** Determine whether Stephen should have the vaccination. **(2 marks)**

E **3** Jemima is going to buy a used car. She is trying to choose between car *X* and car *Y*. The cars are on sale for the same price and both need a service.

The cost of a service for car *X* is £300. There is a 90% chance that there will then be no further maintenance costs for the year. There is a 10% chance of further maintenance costs of £100. If she doesn't service the car there is a 50% chance that there will then be no maintenance costs for the year, and a 50% chance that garage costs will amount to £700.

The cost of a service for car *Y* is £400. There is an 95% chance that there will then be no further maintenance costs for the year. There is a 5% chance of further maintenance costs of £200. If she doesn't service the car there is a 40% chance that there will then be no further maintenance costs for the year, and a 60% chance that garage costs will amount to £800.

**a** Draw a decision tree to show the decisions, outcomes, pay-offs and EMVs. **(6 marks)**

**b** Describe Jemima's best strategy to minimise her maintenance costs. **(2 marks)**

E **4** Alice and Kwame are planning to sell their house. They estimate that, without carrying out any improvements, there is a 60% chance that they could sell for £250 000 and a 40% chance that they could sell for £260 000. By updating the kitchen, they estimate that there is a 70% chance that they could sell for £255 000 and a 30% chance that they could sell for £270 000.

There is an 80% chance that a new kitchen would cost £9000 and a 20% chance that it would cost £10 500.

**a** Draw a decision tree to represent the situation. **(6 marks)**

**b** Describe Alice and Kwame's best strategy for maximising the EMV on the sale of their house. **(2 marks)**

E **5** You have responsibility for a project. If the project is completed on time, then you will be paid a bonus of £30 000. However, if the project is not completed on time then you will have to pay a fine of £1000 per day. You need to choose between two sub-contractors.

Contractor A provides a quote of £260 000 to complete the work. There is some doubt about this contractor's reliability and it is estimated that there is a 40% chance that it will be late by 20 days.

Contractor B provides a quote of £300 000 to complete the work. This contractor has a proven track record of completing projects on time. It is estimated that there is a 5% chance that it will be late by 10 days.

**a** Draw a decision tree to model this situation. **(6 marks)**

**b** Use the EMVs to choose between the contractors. **(2 marks)**

A E **6** A game uses two ordinary six-sided dice.

A player pays £1 to play the game. The dice are thrown and the scores are added together. The player wins £2 and has their £1 returned if the total score is 7, but otherwise they lose.

If a player wins, they can choose to throw again without collecting their winnings from the first throw. This time, if the total score is 7, the player's winnings increase to £10, with their initial £1 returned, otherwise they lose.

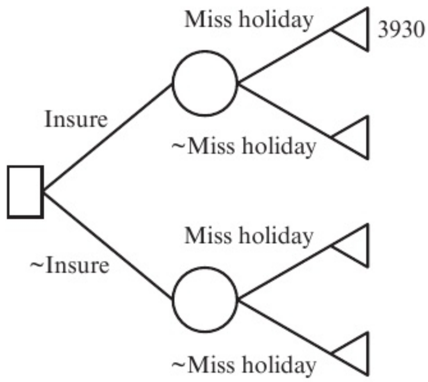
Beth is deciding whether to play the game.

**a** Draw a decision tree to model the decisions, possible outcomes and EMVs. **(6 marks)**

**b** Determine Beth's optimal strategy. **(2 marks)**

E/P **7** A couple has paid £2800 for a holiday. They still have £1200 remaining out of their holiday budget, which they will spend when they are on holiday. They consider taking out insurance that will cover the amount already spent, should they not be able to go on holiday due to illness. Statistics show that there is a probability of 0.004 that illness will mean that they miss their holiday.

**a** Assume that the insurance costs £70. Copy and complete the decision tree and use the EMV of their total budget to determine whether they should take out the insurance. **(6 marks)**




**b** A different insurance company offers the same cover for just £10. Explain how this would influence their decision. **(2 marks)**

**c** State the maximum amount that the couple should pay for insurance using the EMV criterion. **(2 marks)**

E/P **8** Aina and Ben are considering whether to move house to one with more space, or to extend their existing property. Having checked the local property market and obtained quotes from builders, they produce the following table of costs.

| Extend | | Move | |
|---|---|---|---|
| **Cost** | **Probability** | **Cost** | **Probability** |
| £50k | 0.6 | £40k | 0.4 |
| £60k | 0.3 | £70k | 0.4 |
| £70k | 0.1 | £90k | 0.2 |

**a** Draw a decision tree and show, using the EMV of the costs to extend or move, that the better option is to extend their property. **(3 marks)**

A An announcement is made in the local press that a new housing development is to be built in the area and that improvements are to be made to local services. Opinion is divided as to how this will affect the value of existing properties.

Aina and Ben carry out a survey of local estate agents and produce a modified table of possible costs to move.

| Change in value of existing properties | | Cost of moving | | | | | |
|---|---|---|---|---|---|---|---|
| **Change** | **Probability** | **Cost** | **Probability** | **Cost** | **Probability** | **Cost** | **Probability** |
| Reduce | 0.6 | £35k | 0.5 | £68k | 0.3 | £84k | 0.2 |
| None | 0.2 | £50k | 0.6 | £60k | 0.3 | £70k | 0.1 |
| Increase | 0.2 | £45k | 0.2 | £74k | 0.5 | £95k | 0.3 |

**b** Draw a new decision tree to help Aina and Ben decide whether to extend or move. **(7 marks)**

## 8.2 Utility

There are situations in which using the EMV criterion as a basis for making decisions can lead to unfortunate results. To illustrate the point, imagine the following scenario. You are Head of Maths at a school that is trying to raise some much-needed money by holding a summer fete. You are asked to devise a game that will attract a lot of interest and raise a significant amount of money. You propose running a dice game in which six dice are thrown. The charge will be £1 for each attempt and the prize offered will be £2000 for obtaining six sixes in a single throw.

You present the Headteacher with the following argument.

On each throw:

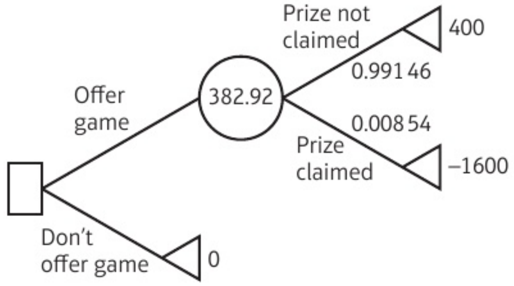
$$\text{P(Win)} = \frac{1}{6^6} = \frac{1}{46\,656}$$

$$\text{P(Not win)} = \frac{46\,655}{46\,656}$$

You expect that there will be 400 attempts during the day:

$$\text{P(Someone wins)} = 1 - \left(\frac{46\,655}{46\,656}\right)^{400} = 0.008\,54$$

The situation may be represented by the following decision tree.



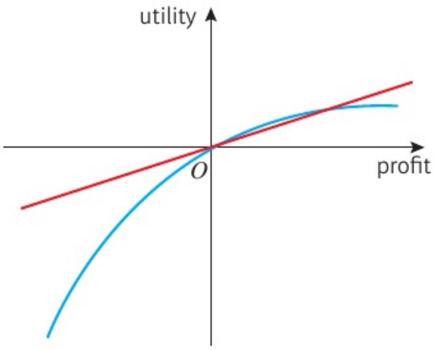

This decision tree assumes that there are 400 attempts, and that at most one prize can be awarded. If more than one person throws six sixes, then a winner would be chosen at random at the end of the day.

A An analysis using the EMV criterion suggests that the school should offer the game. The EMV indicates the average expected return. If you were able to run the game on, say, 100 occasions then you would expect to make a profit of 100 × £382.92 = £38 292.

However, the potential **risk** to the school is very high. On a **single occasion**, the joy of potentially making £400 would evaporate very quickly if someone actually won the £2000. In situations where an average return is not appropriate, and the risk of a loss is to be avoided, a different measure should be used in place of the EMV. This measure is called the **expected utility**; it is written as E(U) and depends on the definition of a **utility function**. A utility function can take various forms and may be tailored to reflect the degree to which an organisation is prepared to take risks.

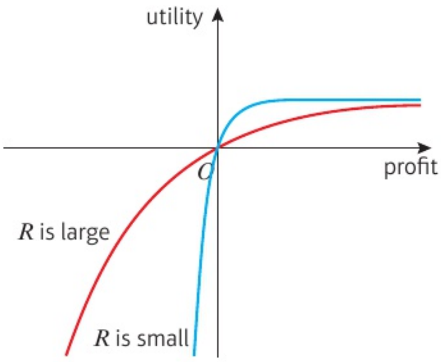
Utility functions designed for 'risk-averse' organisations tend to increase slowly with potential profits, but reduce quickly with potential loss.

The diagram illustrates the difference in behaviour between a linear function (shown in red) and a utility function where risks are to be minimised (shown in blue).




One form that a utility function can take is $u(x) = 1 - e^{\frac{-x}{R}}$ where $x$ is the profit and $R > 0$.

$R$ is a parameter that may be adjusted so that the utility function responds to changes in profit in a way that models the desired behaviour.




**Note** $R$ controls the curvature of the utility function. Smaller values of $R$ increase the curvature and model an increased aversion to risk.

- **A utility function is a function of the pay-offs in a decision tree. It determines the relative value to the individual or organisation of each pay-off.**

**Notation** The utility of a particular pay-off will not usually have units that are meaningful in the context of the question. The utility is therefore measured in **utils**.

## Example 5

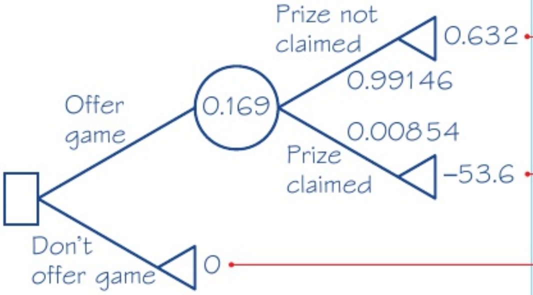
A school has to decide whether to offer the dice game described on pages 252–253. For a pay-off of £$x$, the school determines that its utility function is $u(x) = 1 - e^{-\frac{x}{R}}$.

Calculate the school's expected utility, given that:

**a** $R = 400$ **b** $R = 200$

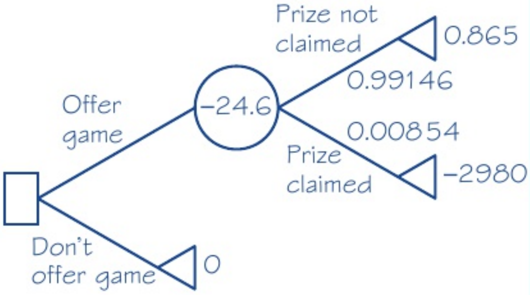
In each case, interpret the results.

**a** The utility function is $u(x) = 1 - e^{-\left(\frac{x}{400}\right)}$




The large value of $R$ has given a small positive value of 0.169 utils for E(U), which indicates that an organisation that is happy to take risks might favour offering the game.

**b** Using $R = 200$




The smaller value of $R$ has produced a negative value of −24.6 utils for E(U), which indicates that an organisation that is not happy to take risks should not offer the game.

Use the utility function to convert the pay-offs to utilities.

The pay-off of 400 converts to $u(400) = 1 - e^{-1} = 0.632$ utils (3 s.f.)

Utility can be negative as well as positive.

The pay-off of −1600 converts to $u(-1600) = 1 - e^{4} = -53.6$ utils (3 s.f.)

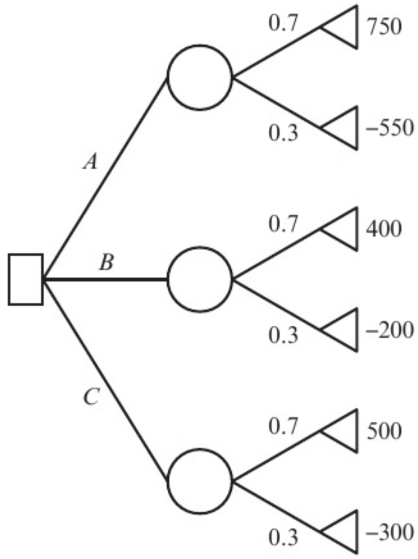
The pay-off of 0 converts to $u(0) = 0$ utils.

**Watch out** You only use the utility function to convert **pay-offs**. You need to calculate the expected utility E(U) at each chance node using the utilities you have just found.

$$E(U) = 0.991\,46 \times 0.632 + 0.008\,54 \times (-53.6)$$
$$= 0.169 \text{ utils (3 s.f)}$$

**Example 6**

A company can choose one of three business options $A$, $B$ or $C$. The profits in pounds associated with each outcome are shown on the decision tree below.

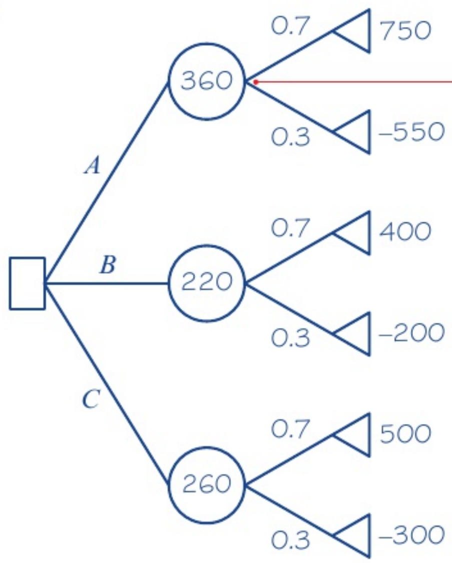



**a** Using the decision tree, calculate the optimal EMV to determine the best choice.

For a profit of £$x$, the company's utility is given by $\sqrt{x + 600}$.

**b** Using expected utility as the criterion, determine the optimal choice.

**a**

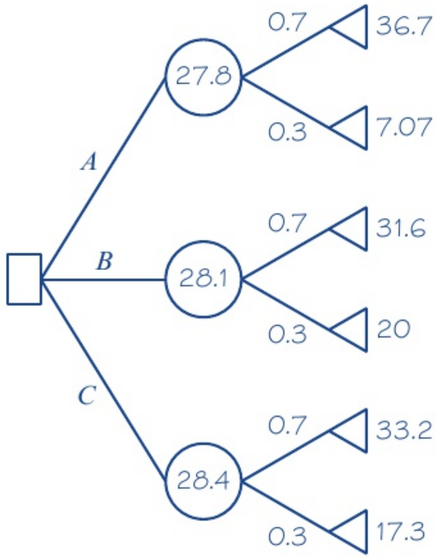



$0.7 \times 750 + 0.3 \times (-550) = 360$

The optimal EMV is £360, which makes option $A$ the best choice using the EMV criterion.

**b** Using the utility function $u(x) = \sqrt{x + 600}$

$u(750) = 36.7$, $u(-550) = 7.07$, $u(400) = 31.6$,
$u(-200) = 20$, $u(500) = 33.2$, $u(-300) = 17.3$




The optimal expected utility is 28.4 utils, which makes option $C$ the best choice using expected utility as the criterion.

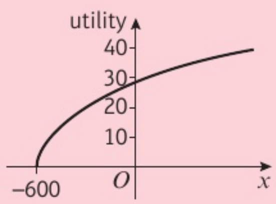
Write the utility as a function.

Rewrite the decision tree, replacing each pay-off with the corresponding utility. Calculate the expected utilities at each chance node.

**Problem-solving**

A sketch of the utility function shows how its value changes rapidly near $x = -600$.

This explains why option $A$ is no longer the favoured choice. The function is designed to represent an aversion to negative pay-offs approaching $-600$.




## Example 7

Tim is considering whether to play a card game. If he picks a Jack, Queen or King, at random from a standard pack of playing cards, then he wins a prize. If he picks any other card, then he loses £5.

**a** Draw a decision tree and state Tim's best course of action using the EMV criterion, given that the prize on offer is £12.

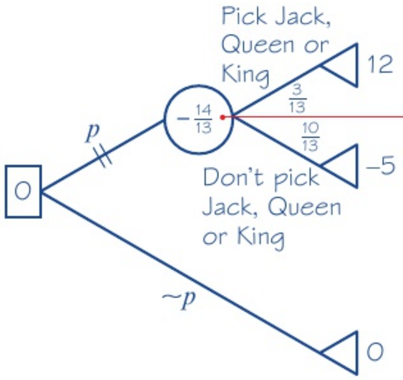
Tim only has £5. He uses utility = $\sqrt[3]{\text{money}^2}$ to reflect his risk aversion.

**b** Calculate the minimum prize amount that would make it worthwhile for Tim to play the game.

**Watch out** The utility function is defined in terms of the total amount of money that Tim has, not in terms of his winnings (or losses).

A

**a**




Using the EMV criterion, Tim should not play the game.

The EMV of the choice to play is negative.

**b** Let the prize amount be £$x$.

If Tim plays the game and wins, he will then have £$(x + 5)$.

If he plays and loses, he will have £0.

If he chooses not to play, then he will still have £5.

Work out the amount of money Tim will have left in the event of each pay-off. Use this to calculate the utility of each pay-off.

The corresponding utility values are $\sqrt[3]{(x+5)^2}$, 0 and $\sqrt[3]{25}$.

The expected utility of playing is $\frac{3}{13} \times \sqrt[3]{(x+5)^2}$

To make it worthwhile for Tim to play the game, we require

$$\frac{3}{13} \times \sqrt[3]{(x+5)^2} > \sqrt[3]{25}$$

$$(x+5)^2 > \left(\frac{13}{3}\right)^3 \times 25$$

$$x > \sqrt{\left(\frac{13}{3}\right)^3} \times 5 - 5$$

$$x > 40.1 \text{ (3 s.f.)}$$

The winning amount should be at least £40.10 to make it worthwhile for Tim to play.

**Problem-solving**

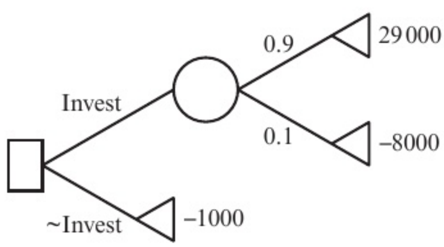
You need to find the minimum value of $x$ such that the expected utility of the decision to play is non-negative.

Use the expected utility to form an inequality, then solve it to find the minimum value of $x$.

## Exercise 8B

**1** A company has paid £1000 for a feasibility study regarding possible investment in a project. The decision tree shows the expected pay-off in pounds.

Use the utility function $u(x) = 1 - e^{-\frac{x}{5}}$, where $x$ is the pay-off in £1000s, to show the utilities on a new decision tree and determine whether the investment should be made.



A E **2** The directors of a company have decided to invest in one of two high-risk projects.

Project $A$ has a 90% chance of returning a profit of £860 000, but a 10% chance of making a loss of £360 000.

Project $B$ has an 80% chance of returning a profit of £600 000, but a 20% chance of making a loss of £180 000.

The directors are wary of simply comparing EMVs and decide to use the utility function $u(x) = \sqrt{x + 400}$, where $x$ is the profit in £1000s.

**a** Draw a decision tree and calculate the expected utilities. **(6 marks)**

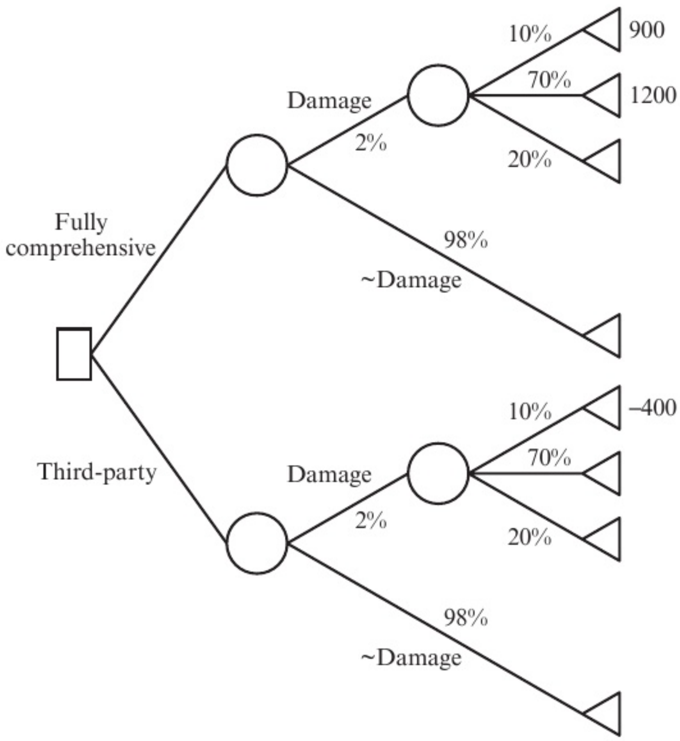
**b** Determine the better option using your answers to part **a**. **(1 mark)**

E/P **3** Sally has bought a used car for £1900. She can pay £400 for third-party insurance, which will not pay for any damage to her own car, or she can pay £700 for fully comprehensive insurance to cover any third-party damage, and damage to her own car up to £1600.

Assume that:

- There is a 2% chance that Sally's car will be damaged in the next year.
- If the car is damaged:
  - There is a 10% chance that it will be damaged beyond repair with £0 value.
  - There is a 70% chance that the cost of repair will be £600.
  - There is a 20% chance that the cost of repair will be £200.
- After the car is repaired, its value remains at £1900.

**a** Copy and complete the decision tree given below.




**Hint** For the first pay-off:
Value of car = 0
Insurance payout = 1600
Insurance cost = –700

For the second pay-off:
Value of car = 1900
Cost of repair = –600
Insurance payout = 600
Insurance cost = –700

**(6 marks)**

**b** Compare the EMVs to determine whether Sally should insure with fully comprehensive insurance, or third-party insurance. **(1 mark)**

A **c** Explain why the use of EMVs may not be the best criterion in this case. **(1 mark)**

Sally decides to apply a utility function to her expected net insurance payout, £$x$. She calculates her utility as $\sqrt{x + 400}$.

**d** Draw a modified decision tree and use the expected utility to determine the type of policy that Sally should take out. **(5 marks)**

E/P **4** Amy has £2 and considers playing a dice game. If she rolls a 5 or a 6 then she wins £$x$; otherwise she loses her £2. Use the function utility = $\sqrt[3]{\text{money}^2}$ to determine the minimum value of $x$ to make it worthwhile for Amy to play the game. **(5 marks)**

## Challenge

Daniel wants to play a lottery game. Tickets cost £1, and the possible winnings, and probability of each prize, are given in the table below.

| | Prize | Probability |
|---|---|---|
| Match 3 | £10 | $\frac{1}{56}$ |
| Match 4 | £50 | $\frac{1}{1032}$ |
| Match 5 | £2000 | $\frac{1}{55\,419}$ |
| Match 5 + bonus ball | £100 000 | $\frac{1}{2\,330\,636}$ |
| Match 6 | £15 000 000 | $\frac{1}{13\,983\,816}$ |

**a** Using Daniel's expected winnings as the criterion, determine whether he should play this game.

In fact, Daniel thinks that nobody needs more than £1 million, so he decides that any winnings in excess of this have no additional value to him. However, he also thinks that each pound of lottery winnings (or losses) up to £50 000 is worth twice as much to him as winnings above this amount.

**b** Define a utility function based on this information.

**c** Use your utility function to determine whether Daniel should play the game.

## Mixed exercise 8

E **1** Jacob is considering playing a game using a standard pack of 52 playing cards. To play the game, a player places a stake of £1 and takes a card from the pack. If the card is a Queen then the player wins £10 and has their stake returned. If the card is a 3, 4 or 5 then a second card is taken from the pack without replacing the first. If this is a Queen then the player wins £5 and has their stake returned. Draw a decision tree and use the EMVs to decide whether Jacob should play the game. **(8 marks)**

E **2** An indemnity insurance company considers providing cover against possible loss to a charity that is offering a roll-a-dice game for a cash prize of £10 000. To win, a player must roll seven sixes in a single throw of seven ordinary dice. The insurance is valid for up to 400 throws and the charge for the cover is £290.

**a** Draw a decision tree for the company and state the EMV of offering the cover. **(8 marks)**

**b** Explain why the EMV criterion is appropriate for the insurance company. **(2 marks)**

A E **3** A game uses two ordinary six-sided dice. A player pays £$x$ for a single throw of the dice. If the total score is more than 10, the player wins £10. If the total score is 5, the player wins £1.

Liam is deciding whether to play the game. Draw a decision tree and calculate the EMV of the game in terms of $x$. State, with a reason, whether Liam should play the game if $x = 1$. **(8 marks)**

E **4** The table shows the possible routes, travel times and possible delay times, in minutes, for a car journey from Stone to Nantwich.

| Main route | Usual time | Possible delay | Probability of delay |
|---|---|---|---|
| A51 | 39 | 10 | 0.1 |
| | | 20 | 0.01 |
| M6 | 35 | 10 | 0.1 |
| | | 30 | 0.01 |
| A500 | 41 | 5 | 0.1 |
| | | 10 | 0.01 |

**a** Draw a decision tree to model the route decisions and possible outcomes. **(8 marks)**

**b** Calculate the minimum expected time and state the corresponding route. **(1 mark)**

**c** Give a reason why the minimum expected time might not correspond to the favoured route. **(2 marks)**

E **5** Omar decides to offer his car for sale priced at £2000 or nearest offer.

An advertising company offers a premium package and a standard package.

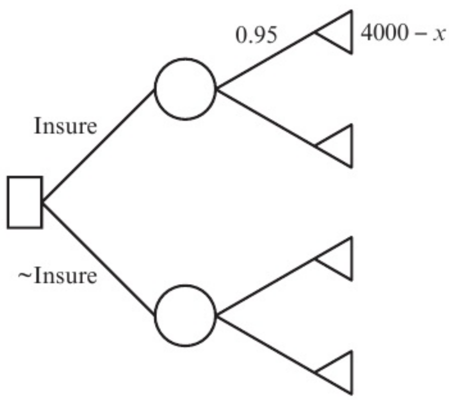
The premium package costs £75. There is a 90% chance that the car will sell in the first week with this package. There is then an 80% chance that the car will sell for £1950 and a 20% chance that it will sell for £1900. If the car doesn't sell in the first week, then there is a 90% chance that it will sell for £1850 and a 10% chance that it will sell for £1800.

The standard package costs £35. There is a 70% chance that the car will sell in the first week with this package. There is then an 80% chance that the car will sell for £1900 and a 20% chance that it will sell for £1850. If the car doesn't sell in the first week, then there is a 70% chance that it will sell for £1800 and a 30% chance that it will sell for £1750.

**a** Draw a decision tree to model this situation. **(10 marks)**

**b** Calculate the optimum EMV and state the corresponding package. **(4 marks)**

E/P **6** Zoe has camera equipment worth £4000. She has been asked to film an outdoor event where the organisers have assured her that there is a 95% chance that her equipment will be safe. However, if things do go wrong, then her equipment will be destroyed.




**a** Copy and complete the decision tree to show the EMV in pounds of the camera equipment, if the cost of the insurance premium is £$x$. **(4 marks)**

**b** Using the EMV criterion, state the maximum amount that Zoe should pay for insurance. **(2 marks)**

A **c** The insurance company argues that expected utility should be used in place of the EMV to reflect the value of the equipment to Zoe. Use utility = $\sqrt[3]{\text{value}}$ to calculate a revised figure for the maximum insurance premium Zoe should pay. **(4 marks)**

E/P **7** Joe is considering whether to play a card game. If he picks a red even-numbered card, at random from a standard pack of playing cards, then he wins £3. If he picks any other card, then he loses £1.

**a** Draw a decision tree and state Joe's best course of action using the EMV criterion. **(6 marks)**

**b** Joe has just £1. Use utility = $\sqrt[3]{\text{money}^2}$ to calculate the minimum winning amount to make it worthwhile for Joe to play the game. **(6 marks)**

E/P **8** **a** Explain the purpose of a utility function in decision analysis. **(2 marks)**

**b** A company with current assets of £70 000 is committed to the completion of a project. The latest analysis indicates that there is a 10% chance of making a loss of £30 000 and a 90% chance of breaking even on the project. The company uses expected utility to assess the maximum amount, £$p$, it should pay in insurance to cover the possible loss. Use $\text{u}(x) = \ln x$, where $x$ represents the total value of the company's assets, to determine the value of $p$. **(8 marks)**

## Challenge

The directors of a company are considering three projects A, B and C.

Project A has a 70% chance of making a profit of £45 000, a 20% chance of making a profit of £60 000 and a 10% chance of making a profit of £80 000.

Project B has a 70% chance of making a profit of £35 000, a 25% chance of making a profit of £70 000 and a 5% chance of making a profit of £120 000.

Project C has a 40% chance of making a profit of £15 000, a 30% chance of making a profit of £50 000 and a 30% chance of making a profit of £150 000.

For a profit of £$x$, the company directors determine that their utility is given by the following function:

$$\text{u}(x) = \begin{cases} x & 0 < x < 50\,000 \\ \dfrac{x}{2} + 25\,000 & 50\,000 < x < 100\,000 \\ 75\,000 & x > 100000 \end{cases}$$

**a** Describe the risk profile that is represented by this utility function.

**b** Name the project that the directors should choose and state their expected utility and their expected profit.

## Summary of key points

A

**1** In a **decision tree**:
- boxes represent **decision nodes**
- triangles represent **end** (or **pay-off**) **nodes**
- circles represent **chance nodes**

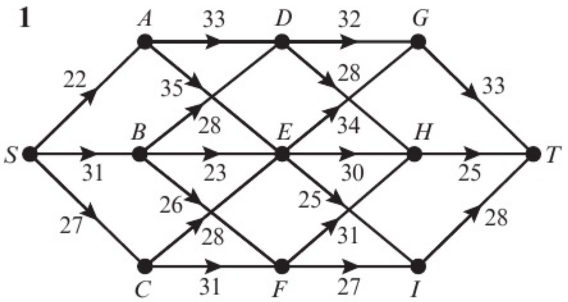
**2** You annotate the decision tree to show the following information:
- For an event, the probabilities of each outcome are given on the branches.
- The pay-off for each end node is written next to that node.
- The expected value of each event is written inside the chance node for that event. This value is called the **expected monetary value (EMV)**.

**3** An **optimal strategy** based on the EMV criterion is a set of decisions chosen so as to optimise the EMV.

**4** A **utility function** is a function of the pay-offs in a decision tree. It determines the relative value to the individual or organisation of each pay-off.

# Review exercise 2

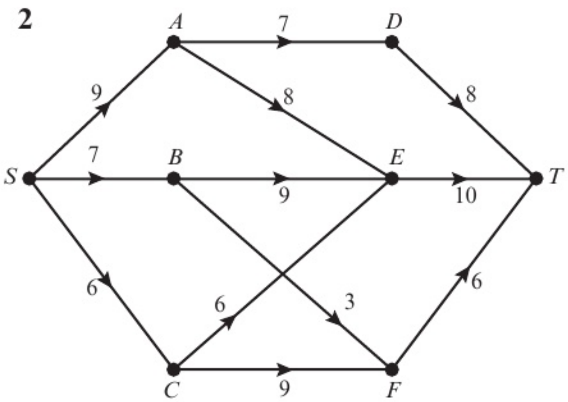
E 1




Use dynamic programming to find the shortest route from $S$ to $T$. State the route and its length. **(10)**

← Section 5.1

E 2




The network shows possible routes that an aircraft can take from $S$ to $T$. The numbers on the directed arcs give the amount of fuel used on that part of the route, in tonnes. The airline wishes to choose the route for which the maximum amount of fuel used on any part of the route is as small as possible.

**a** Write down the type of dynamic programming problem that needs to be solved. **(9)**

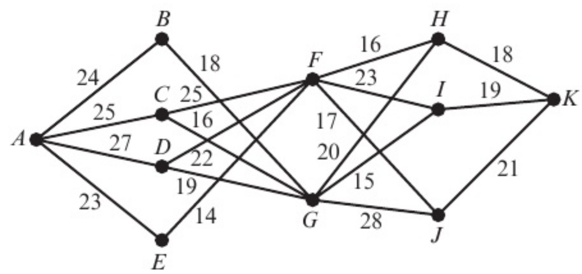
**b** Use dynamic programming to find the optimum route. **(3)**

**c** Given that the aircraft can refuel at each vertex, state the minimum required fuel capacity for this aircraft to travel from $S$ to $T$. **(1)**

**d** Give a reason why it may not be advisable for an aircraft with exactly this fuel capacity to attempt this journey. **(1)**

← Section 5.2

E/P **3 a** Explain what is meant by a maximin route in dynamic programming, and give an example of a situation that would require a maximin solution. **(2)**



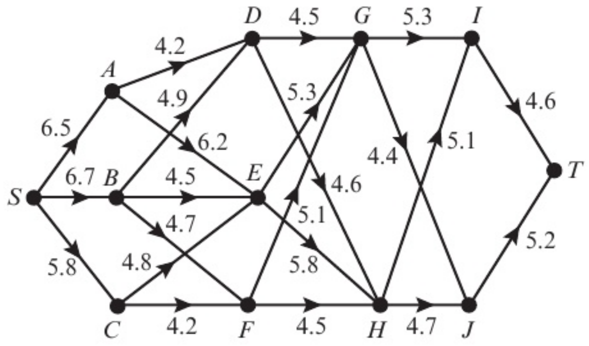

A maximin route is to be found through the network shown.

**b** Use dynamic programming to find the optimum route. **(9)**

**c** List **all** other maximin routes through the network. **(2)**

← Section 5.2

A E **4** The diagram represents a network of country lanes. The number on each arc represents the width of the corresponding lanes in metres.




A large truck is positioned at $S$ and the driver has to find a route to a farm at $T$ avoiding narrow lanes as far as possible.

**a** Solve a maximin dynamic programming problem to find two possible routes. **(11)**

**b** State the minimum lane width on each route. **(1)**

**c** It is proposed to widen the lane $HJ$ by 2 metres. Determine the best route if this proposal is put into action. State the new minimum lane width for the route. **(2)**

← Section 5.2

E/P **5** An engineering firm makes motors. They can make up to five in any one month, but if they make more than four they have to hire additional premises at a cost of £500 per month. They can store up to two motors for £100 per motor per month. The overhead costs are £200 in any month in which work is done.

Motors are delivered to buyers at the end of each month. There are no motors in stock at the beginning of May and there should be none in stock after the September delivery.

The order book for motors is:

| **Month** | May | June | July | Aug. | Sept. |
|---|---|---|---|---|---|
| **Number of motors** | 3 | 3 | 7 | 5 | 4 |

A Use dynamic programming to determine the production schedule that minimises the costs, showing your working in a table. **(12)**

← Section 5.3

E/P **6** Kris produces custom made racing cycles. She can produce up to four cycles each month, but if she wishes to produce more than three in any one month she has to hire additional help at a cost of £350 for that month. In any month when cycles are produced, the overhead costs are £200. A maximum of three cycles can be held in stock in any one month, at a cost of £40 per cycle per month. Cycles must be delivered at the end of the month. The order book for cycles is

| **Month** | Aug | Sept | Oct | Nov |
|---|---|---|---|---|
| **Number of cycles required** | 3 | 3 | 5 | 2 |

Disregarding the cost of parts and Kris's time,

**a** determine the total cost of storing two cycles and producing four cycles in a given month, making your calculations clear. **(2)**

There is no stock at the beginning of August and Kris plans to have no stock after the November delivery.

**b** Use dynamic programming to determine the production schedule which minimises the costs, showing your working in a table. **(9)**

The fixed cost of parts is £600 per cycle and of Kris's time is £500 per month. She sells the cycles for £2000 each.

**c** Determine her total profit for the four-month period. **(1)**

← Section 5.3

E/P **7** Joan sells ice cream. She needs to decide which three shows to visit over a three-week period in the summer. She starts the

A

three-week period at home and finishes at home. She will spend one week at each of the three shows she chooses, travelling directly from one show to the next.

Table 1 gives the week in which each show is held. Table 2 gives the expected profits from visiting each show. Table 3 gives the cost of travel between shows.

Table 1

| **Week** | 1 | 2 | 3 |
|---|---|---|---|
| **Shows** | *A*, *B*, *C* | *D*, *E* | *F*, *G*, *H* |

Table 2

| **Show** | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| **Expected profit (£)** | 900 | 800 | 1000 | 1500 |

| **Show** | *E* | *F* | *G* | *H* |
|---|---|---|---|---|
| **Expected profit (£)** | 1300 | 500 | 700 | 600 |

Table 3

| **Travel costs (£)** | *A* | *B* | *C* | *D* |
|---|---|---|---|---|
| **Home** | 70 | 80 | 150 | |
| ***A*** | | | | 180 |
| ***B*** | | | | 140 |
| ***C*** | | | | 200 |
| ***D*** | | | | |
| ***E*** | | | | |

| **Travel costs (£)** | *E* | *F* | *G* | *H* |
|---|---|---|---|---|
| **Home** | | 80 | 90 | 70 |
| ***A*** | 150 | | | |
| ***B*** | 120 | | | |
| ***C*** | 210 | | | |
| ***D*** | | 200 | 160 | 120 |
| ***E*** | | 170 | 100 | 110 |

It is decided to use dynamic programming to find a schedule that maximises the total expected profit, taking into account the travel costs.

**a** Define suitable stage, state and action variables. **(3)**

**b** Determine the schedule that maximises the total profit. Show your working in a table. **(10)**

**c** Advise Joan on the shows that she should visit and state her total expected profit. **(2)**

← Section 5.3

E **8** A two-person zero-sum game is represented by the following pay-off matrix for player *A*.

| | ***B* plays 1** | ***B* plays 2** | ***B* plays 3** | ***B* plays 4** |
|---|---|---|---|---|
| ***A* plays 1** | −4 | −5 | −2 | 4 |
| ***A* plays 2** | −1 | 1 | −1 | 2 |
| ***A* plays 3** | 0 | 5 | −2 | −4 |
| ***A* plays 4** | −1 | 3 | −1 | 1 |

**a** Explain what the term ‘play-safe strategy’ means in game theory. **(2)**

**b** Determine the play-safe strategy for each player, and write down the pay-off for player *A* if both players adopt their play-safe strategies. **(3)**

**c** Show that this game has a stable solution. **(2)**

**d** State the value of the game to player *B*. **(1)**

← Section 6.1

E **9** Emma and Freddie play a zero-sum game. This game is represented by the following pay-off matrix for Emma.

| | ***F* plays 1** | ***F* plays 2** | ***F* plays 3** |
|---|---|---|---|
| ***E* plays 1** | −4 | −1 | 3 |
| ***E* plays 2** | 2 | 1 | −2 |

**a** Explain what is meant by a zero-sum game. **(2)**

**b** Show that this game has no stable solution. **(2)**

**c** Find Emma’s optimal mixed strategy, and the value of the game to her. **(5)**

**d** State the value of the game to Freddie and write down his pay-off matrix. **(3)**

← Section 6.3

E **10** A two-person zero-sum game is represented by the following pay-off matrix for player $A$.

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | 5 | −2 |
| *A* plays 2 | −2 | 3 |
| *A* plays 3 | 4 | 0 |
| *A* plays 4 | 2 | −1 |

Supraj says that the 5 in the top-left cell is a saddle point because it is the largest value in its column.

**a** Give a reason why Supraj is wrong. **(1)**

**b** Verify that this game has no stable solution. **(2)**

**c i** Find the best strategy for player $B$, defining any variables you use.

**ii** Find the value of the game to player $A$. **(8)**

← Sections 6.1, 6.2, 6.3

E/P **11** Amir and Becky are playing a zero-sum game. Each player has two cards, numbered 4 and 5. The players each choose one of their cards, and turn it face up.

- If the sum of the numbers showing is **odd** then Amir wins the **product** of the two numbers showing in pounds.
- If the sum of the numbers showing is **even** than Becky wins the **product** of the two numbers in pounds.

For example, if Amir plays 5 and Becky plays 4 then the total is 9. This is odd, so Amir wins $5 \times 4 = £20$.

**a** Draw the pay-off matrix for this game, from Amir's point of view. **(2)**

**b** Verify that there is no stable solution and find Amir's optimal mixed strategy. **(4)**

**c** State the value of the game to Amir. **(1)**

← Sections 6.1, 6.3

A E **12** Andrew ($A$) and Barbara ($B$) play a zero-sum game. This game is represented by the following pay-off matrix for Andrew.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 3 | 5 | 4 |
| *A* plays 2 | 1 | 4 | 2 |
| *A* plays 3 | 6 | 3 | 7 |

**a** Explain why this matrix may be reduced to

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | 3 | 5 |
| *A* plays 3 | 6 | 3 |

**(1)**

**b** Hence find the best strategy for each player and the value of the game to player $A$. **(5)**

← Section 6.2

E **13** A two person zero-sum game is represented by the following pay-off matrix for player $A$.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 2 | −1 | 3 |
| *A* plays 2 | 1 | 3 | 0 |
| *A* plays 3 | 0 | 1 | −3 |

**a** Identify the play-safe strategies for each player. **(4)**

**b** Verify that there is no stable solution to this game. **(1)**

**c** Explain why the pay-off matrix above may be reduced to

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 2 | −1 | 3 |
| *A* plays 2 | 1 | 3 | 0 |

**(1)**

**d** Find player $A$'s optimal mixed strategy, and the value of the game to player $A$. **(6)**

← Sections 6.2, 6.3

A E/P **14** **a** Explain the difference between a pure strategy and a mixed strategy in a two-person zero-sum game. **(2)**

A two-person zero-sum game is represented by the following pay-off matrix for player $A$.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 0 | 1 | –1 |
| *A* plays 2 | 4 | –3 | 2 |
| *A* plays 3 | 2 | –1 | 0 |

**b** Reduce the matrix so that one player has only two choices. **(1)**

**c** Verify that the game has no stable solution. **(2)**

**d** Use your reduced matrix to find the best strategy for the player with only 2 choices. **(5)**

← Sections 6.2, 6.3

E **15** A two person zero-sum game is represented by the following pay-off matrix for player $A$.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | –3 | 2 | 5 |
| *A* plays 2 | 4 | –1 | –4 |

**a** Write down the pay-off matrix for player $B$. **(2)**

**b** Formulate the game as a linear programming problem for player $B$, writing the constraints as equalities and stating your variables clearly. **(5)**

← Section 6.4

A E **16** A two-person zero-sum game is represented by the following pay-off matrix for player $A$.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 | *B* plays 4 |
|---|---|---|---|---|
| *A* plays 1 | –2 | 1 | 3 | –1 |
| *A* plays 2 | –1 | 3 | 2 | 1 |
| *A* plays 3 | –4 | 2 | 0 | –1 |
| *A* plays 4 | 1 | –2 | –1 | 3 |

**a** Verify that there is no stable solution to this game. **(3)**

**b** Explain why the 4 × 4 game above may be reduced to the following 3 × 3 game. **(1)**

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | –2 | 1 | 3 |
| *A* plays 2 | –1 | 3 | 2 |
| *A* plays 4 | 1 | –2 | –1 |

**c** Formulate the 3 × 3 game as a linear programming problem for player $A$. Write the constraints as inequalities. Define your variables clearly. **(5)**

← Sections 6.1, 6.2, 6.3, 6.4

E **17** **a** Explain briefly what is meant by a zero-sum game. **(1)**

A two-person zero-sum game is represented by the following pay-off matrix for player $A$.

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | 5 | 2 | 3 |
| *A* plays 2 | 3 | 5 | 4 |

**b** Verify that there is no stable solution to this game. **(2)**

**c** Find the best strategy for player $A$ and the value of the game to her. **(2)**

**d** Formulate the game as a linear programming problem for player $B$. Write the constraints as inequalities and define your variables clearly. **(5)**

← Sections 6.1, 6.3, 6.4

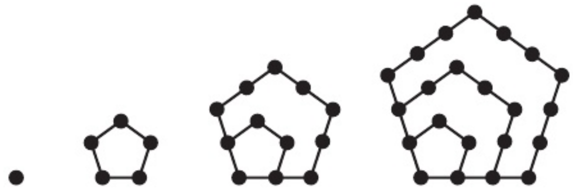
E **18** **a** Find the general solution of the recurrence relation

$$u_n = 4u_{n-1} + 1, n \geqslant 1 \quad \textbf{(4)}$$

**b** Find the particular solution of the recurrence relation given that $u_0 = 7$. **(2)**

← Section 7.2

E/P **19** The diagram shows the first four pentagonal numbers.



$p_1 = 1$ $p_2 = 5$ $p_3 = 12$ $p_4 = 22$

**a** Find $p_5$ and $p_6$. **(1)**

**b** Find, in terms of $p_{n-1}$, a recurrence relation for $p_n$. **(2)**

**c** Solve the recurrence relation to find a closed form for the $n$th pentagonal number. **(3)**

**d** Find $p_{100}$. **(1)**

← Sections 7.1, 7.2

E **20** Solve the recurrence relation

$$u_n = 2u_{n-1} + 3n + 1, \text{ for } n \geqslant 0,$$

where $u_0 = 11$. **(4)**

← Section 7.2

E/P **21** A sequence is defined by the recurrence relation

$$u_{n+1} - 3u_n = 10, \text{ with } u_1 = 7$$

**a** Find $u_3$. **(1)**

**b** **i** Solve the recurrence relation.

**ii** Find the smallest value of $n$ for which $u_n$ is greater than 1 million. **(4)**

← Section 7.2

E/P **22** A hospital patient receives 100 mg of a drug every 4 hours. Once administered, 20% of the drug remaining in the patient's body is lost every hour.

Initially, the drug is not present in the patient's body.

Let $u_n$ represent the amount of the drug in the patient's body immediately after the $n$th dose.

**a** Find, in terms of $u_n$, a recurrence relation for $u_{n+1}$. **(2)**

**b** Solve the recurrence relation for $u_n$. **(3)**

The amount of the drug present in the patient's body must not exceed 160 mg.

**c** What is the maximum number of doses that can be administered in this way? **(3)**

← Sections 7.1, 7.2

A E **23** **a** Find the general solution of the recurrence relation

$$u_{n+2} = 4u_{n+1} + 5u_n, n \geqslant 2 \quad \textbf{(3)}$$

**b** Find the particular solution of the recurrence relation given that

$$u_0 = 8 \text{ and } u_1 = -20. \quad \textbf{(3)}$$

← Section 7.3

E **24** Consider the recurrence relation

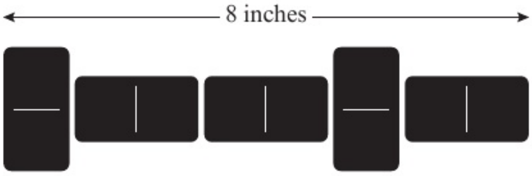
$$3u_{n+2} + 10u_{n+1} - 8u_n = 20$$

**a** Find a constant $k$ such that $u_n = k$ is a particular solution to this recurrence relation. **(2)**

**b** Hence solve the recurrence relation given that $u_0 = 0$ and $u_1 = 1$. **(5)**

← Section 7.3

A E/P **25** Rectangular 2 inch by 1 inch dominoes can be placed in a row horizontally or vertically. The diagram shows one possible row of length 8 inches.



Let $x_n$ represent the total number of different ways that a row of length $n$ inches can be formed.

**a** Explain why $x_n$ satisfies the recurrence relation $x_{n+2} = x_{n+1} + x_n$, $x_1 = 1$, $x_2 = 2$. **(3)**

**b** Find the number of possible ways of forming a row of length 8 inches. **(1)**

**c** **i** By solving the recurrence relation in part **a**, find a closed form for $u_n$.

**ii** Find the number of ways of forming a row of length 2 feet. **(5)**

← Section 7.1, 7.3

E/P **26** A game uses three ordinary dice.

It costs 50p to play the game. A player rolls all three dice at the same time and wins £5 if they each show the same score.

If no two dice show the same score, then the player loses.

If exactly two of the dice show the same score, then the player can choose to take winnings of £1 or continue by rolling one of the dice again. If the player continues and all three scores are the same then the player wins £5, otherwise the player loses.

The original stake of 50p is not returned for any outcome.

**a** Draw a decision tree to model the possible decisions and outcomes. **(6)**

**b** Calculate the optimal EMV and state the optimal strategy indicated by your decision tree. **(2)**

← Section 8.1

A E/P **27** The management team of a company are considering taking on a project. If the project is very successful, then it can earn £75 000 for the company. If it is moderately successful, then it can earn £45 000 for the company. If it is unsuccessful then the company can lose £30 000.

The team estimate that the probability of a very successful outcome is 20% and the probability of a moderately successful outcome is 50%.

The company can choose to hire a business consultant at a cost of £$x$. The consultant can provide advice that will increase the probability of a very successful outcome to 35% and the probability of a moderately successful outcome to 55%.

**a** Draw a decision tree to model the company's choices and outcomes. **(6)**

**b** Write, in terms of $x$, the EMV of the project if the company decides to hire the consultant. **(2)**

**c** Determine the maximum amount that the company should pay to hire the consultant. **(2)**

← Section 8.1

A E/P **28** A game uses a standard pack of 52 cards.

A player pays £2 to play the game and then selects a card at random.

If the card is red and numbered 5, 6 or 7 then the player wins £10 and has the £2 stake returned. If any other red card is selected then the player loses.

If any black card is selected then the player can either forfeit the stake or pay a further £1 to continue.

If the player continues then a second card is selected at random after replacing the first card. If this card is black then the total stake of £3 is returned, otherwise the player loses.

**a** Draw a decision tree to model a player's decisions and the possible outcomes. **(6)**

**b** Calculate the EMV of playing the game and describe the optimal strategy. **(2)**

← Section 8.1

E/P **29** The organisers of a village fête have estimated how their expected profit relates to the weather conditions on the day.

**Fine conditions**

| **Profit** | £5000 | £4000 | £3000 |
|---|---|---|---|
| **Chance** | 70% | 20% | 10% |

They estimate that light rain will reduce the profit by 10% in each case and that heavy rain will reduce the profit by 40% in each case.

The organisers are considering two possible dates for the fête. One is the first Saturday in April and the other is the last Saturday in April.

The forecast for the first Saturday is 50% chance of fine weather, 30% chance of light rain and 20% chance of heavy rain.

The forecast for the second Saturday is 40% chance of fine weather, 48% chance of light rain and 12% chance of heavy rain.

Calculate the EMV of the profits for each of the two possible dates and determine the date that the organisers should choose. **(8)**

← Section 8.1

E/P **30** Ben is considering playing a game using two ordinary dice. The dice are rolled and the scores added. A total score of 10 or more wins a prize of £10. It costs £1 to play the game and the stake is not returned.

**a** Draw a decision tree and use the EMV criterion to determine whether Ben should play. **(4)**

Ben only has £1. He decides to use the utility function

$$\text{utility} = \sqrt[3]{\text{money}^2}$$

**b** Use expected utility to decide whether or not Ben should play. **(6)**

← Section 8.2

E/P **31** A company is deciding whether to invest in project $X$ or project $Y$.

Project $X$ has an 85% chance of returning a profit of £720 000, but a 15% chance of making a loss of £240 000.

Project $Y$ has a 75% chance of returning a profit of £500 000, but a 25% chance of making a loss of £170 000.

The directors use the utility function

$$u(x) = \sqrt{x + 300}$$

where $x$ is the profit in £1000s.

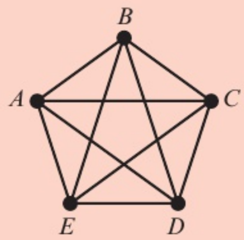
**a** Draw a decision tree and calculate the expected utilities. **(8)**

**b** Determine the better option using expected utility as the criterion, and find the expected profit in this case. **(3)**

← Section 8.2

## Challenge

**1** The diagram shows the complete graph with five vertices, $K_5$.




**a** Explain why the total number of different walks of length $n$ on $K_5$ starting at vertex $A$ is $4^n$.

A **closed walk** on a graph starts and ends at the same vertex. So, for example

$ABDA$ is a closed walk of length 3

$CBC$ is a closed walk of length 2

Let $u_n$ be the total number of possible closed walks of length $n$ on $K_5$, which start and end at $A$.

**b** Find a recurrence relation for $u_n$ in terms of $u_{n-1}$, and justify your answer.

**c** Hence, or otherwise, show that

$$u_n = \frac{4^n + 4(-1)^n}{5}$$

**d** Prove that the total number of closed walks of length $n$ on $K_p$ is

$$(p-1)^n + (p-1)(-1)^n$$

← D1 Chapter 2, D2 Chapter 7

**2** A two-person zero-sum game is represented by the following pay-off matrix:

| | $B$ plays 1 | $B$ plays 2 | $B$ plays 3 |
|---|---|---|---|
| $A$ plays 1 | 2 | 1 | −1 |
| $A$ plays 2 | 3 | −2 | 5 |
| $A$ plays 3 | 1 | 2 | 2 |

Given that in $A$'s best strategy, she plays 3 with probability $\frac{2}{3}$,

**a** find player $A$'s complete best strategy.

Give further that, when playing his best strategy, $B$'s expected losses will be minimised when $A$ plays 1,

**b** find player $B$'s complete best strategy.

← Chapter 6

A

**3** Louise has £10 000 and is considering two investment opportunities.

The first one has a 60% chance of adding 12% to the amount invested, but a 40% chance of adding only 5% to the amount invested.

The second one has a 70% chance of adding 20% to the amount invested, but a 30% chance of losing $p$% of the amount invested.

Louise uses utility = ln ($x$), where $x$ is the total amount of money that Louise has at the end of the investment period, to choose between the two investment opportunities.

Find the maximum value of $p$ such that the second option is more favourable.

← Section 8.2

# Exam-style practice

## Further Mathematics AS Level Decision Mathematics 2

**Time: 50 minutes**
**You must have: Mathematical Formulae and Statistical Tables, Calculator**

**1** Five workers $A$, $B$, $C$, $D$ and $E$ are to be assigned to five tasks 1, 2, 3, 4 and 5.

Each worker must be assigned to just one task and each task is to be completed by just one worker.

The table shows the profit made by assigning each worker to each task.

| | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| ***A*** | 51 | 47 | 62 | 50 | 55 |
| ***B*** | 54 | 51 | 60 | 53 | 51 |
| ***C*** | 49 | 52 | 58 | 55 | 53 |
| ***D*** | 52 | 56 | 61 | 58 | 57 |
| ***E*** | 56 | 48 | 59 | 55 | 56 |

Reducing rows first, use the Hungarian algorithm to obtain an allocation that maximises the total profit. Show the table at each stage and state the maximum profit. **(9)**

**2** The table shows the pay-offs for a two-person zero-sum game.

| | ***Y* plays 1** | ***Y* plays 2** | ***Y* plays 3** | ***Y* plays 4** |
|---|---|---|---|---|
| ***X* plays 1** | 3 | −1 | 2 | 1 |
| ***X* plays 2** | 1 | 2 | 4 | 3 |

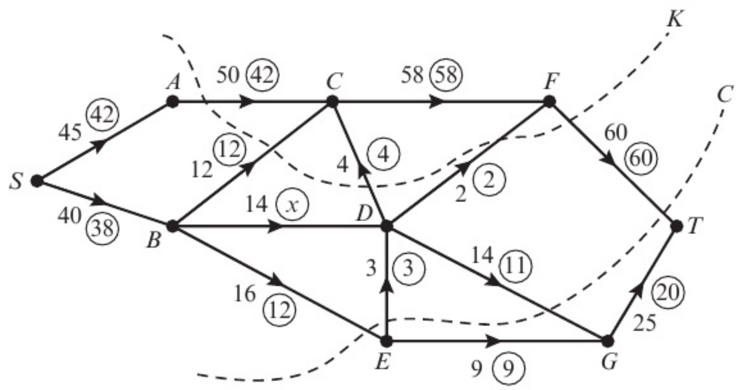
**a** Explain what is meant by a 'zero-sum' game. **(1)**

**b** Show that there is no stable solution. **(2)**

**c** Find the best strategy for player $X$ and the value of the game to him. **(9)**

**3** The diagram represents a capacitated directed network of water pipes.

The numbers on the arcs represent the capacities of the pipes and the numbers in circles represent a feasible flow.




**a** Explain why $K$ is not a cut. **(1)**

**b** Write down:

**i** the capacity of the cut $C$

**ii** the value of $x$

**iii** the value of the flow. **(3)**

**c** Find a flow-augmenting path that will increase the value of the flow by 3. **(2)**

**d** Prove that the flow is then maximal. **(3)**

**4** An amount, £$X$, is taken out as a loan to buy a property. At the end of each month, interest of $r$% is added to the balance of the loan, and a fixed amount, £$p$, is then repaid.

Let $L_n$ be the balance of the loan after $n$ payments have been made.

**a** Write a first-order recurrence relation for $L_n$. **(1)**

**b** Show that $L_n = \dfrac{100p}{r}$ is a particular solution to this recurrence relation. **(2)**

**c** Solve the recurrence relation for $L_n$. **(4)**

A customer wishes the loan to be fully repaid after $n$ months.

**d** Show that the customer's minimum monthly payment should be $\dfrac{Xr}{100 - 100^{n+1}(100 + r)^{-n}}$ **(3)**

# Exam-style practice

## Further Mathematics
## A Level
## Decision Mathematics 2

**Time: 1 hour 30 minutes**
**You must have: Mathematical Formulae and Statistical Tables, Calculator**

1 **a** Find the general solution of the recurrence relation $u_{n+2} - 6u_{n+1} + 9u_n = 0$ **(3)**

Given that $u_1 = \frac{21}{4}$ and $u_2 = -\frac{93}{4}$,

**b** find the particular solution of the recurrence relation $u_{n+2} - 6u_{n+1} + 9u_n = 15$ **(6)**

2 The pay-off matrix for a zero-sum game between *X* and *Y* is shown below.

| | ***Y* plays 1** | ***Y* plays 2** | ***Y* plays 3** |
|---|---|---|---|
| ***X* plays 1** | 3 | 2 | 1 |
| ***X* plays 2** | 1 | 4 | 2 |
| ***X* plays 3** | −2 | 2 | 1 |
| ***X* plays 4** | 1 | 5 | 4 |

**a** Use dominance arguments to reduce the pay-off matrix so that one player has only two choices. **(2)**

**b** Verify that *X* and *Y* should play mixed strategies. **(2)**

**c** Use your reduced pay-off matrix to find the optimum mixed strategy for the player with two choices and state the value of the game to her. **(7)**

3 Four workers *P*, *Q*, *R* and *S* are to be assigned to four tasks *W*, *X*, *Y* and *Z*. Each worker can only be assigned to one task and each task is to be completed by just one worker.

The table shows the cost, in £s, of allocating each worker to each task.

| | ***W*** | ***X*** | ***Y*** | ***Z*** |
|---|---|---|---|---|
| ***P*** | 35 | 41 | 52 | 47 |
| ***Q*** | 43 | 45 | 48 | 46 |
| ***R*** | 37 | 48 | 45 | 50 |
| ***S*** | 42 | 50 | 44 | 48 |

The Hungarian algorithm is to be used to allocate workers to tasks so that the total cost is minimised.

**a** Reducing rows first, use the Hungarian algorithm to find the optimum allocation showing the table used at each stage. State the minimum cost. **(7)**

**b** Formulate the problem as a linear programming problem. You should define your decision variables, objective function and constraints. **(5)**

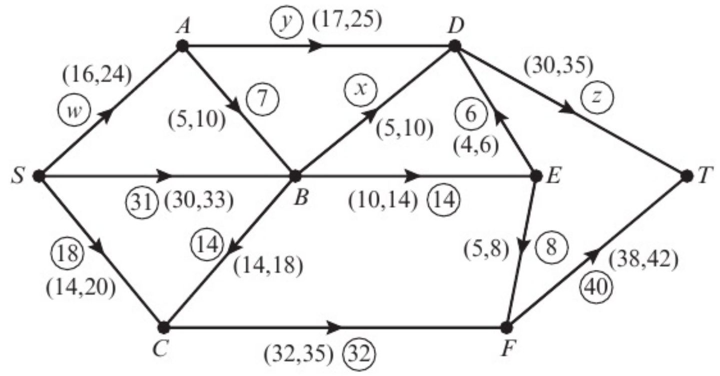
**4** Three warehouses $A$, $B$ and $C$ supply three shops $P$, $Q$ and $R$ with pallets of frozen cakes. The table shows the transportation costs, in £s per pallet, from each warehouse to each shop. The table also shows the supply available at each warehouse and the demand required at each shop.

| | ***P*** | ***Q*** | ***R*** | **Supply** |
|---|---|---|---|---|
| ***A*** | 11 | 12 | 15 | 10 |
| ***B*** | 14 | 10 | 14 | 16 |
| ***C*** | 12 | 16 | 13 | 14 |
| **Demand** | 12 | 14 | 14 | |

**a** Use the north-west corner method to find an initial solution. **(1)**

**b** Use the transportation algorithm to solve the problem, making your method clear. Explain how you know that your solution is optimal. **(11)**

**5** The diagram shows a capacitated directed network. The numbers on each arc represent lower and upper capacities. The numbers in circles represent an initial flow.




**a** Find the values of $w$, $x$, $y$ and $z$. **(3)**

**b** Find the value of the cut through

**i** $AD$, $BD$, $BE$, $CF$

**ii** $DT$, $DE$, $BE$, $CF$ **(2)**

**d** Explain what your answers to part **b** show about the maximum flow. **(1)**

**e** Find two flow-augmenting paths to increase the flow by a total of 4. **(4)**

**f** Show that the flow is then maximal. **(2)**

**6** A small start-up company can choose between three projects $A$, $B$ and $C$. It is estimated that:

- Project $A$ has a probability of $\frac{1}{3}$ of making a £20 000 profit, a probability of $\frac{1}{2}$ of making a £10 000 profit and a probability of $\frac{1}{6}$ of making a £20 000 loss.
- Project $B$ has a probability of $\frac{1}{4}$ of making a £15 000 profit, a probability of $\frac{1}{2}$ of making an £8000 profit and a probability of $\frac{1}{4}$ of making a £2000 loss.
- Project $C$ has a probability of $\frac{1}{2}$ of making a £12 000 profit, a probability of $\frac{1}{4}$ of making a £5000 profit and a probability of $\frac{1}{4}$ of making a £1000 loss.

**a** Draw a decision tree to represent the decisions and possible pay-offs. Use the expected monetary value (EMV) criterion to determine which project should be undertaken. **(3)**

**b** Explain why it would be advisable to use a utility function in this case. **(1)**

**c** Draw a new decision tree using the utility function $u(x) = \sqrt{x + 25}$, where $x$ is the profit in £1000s. **(3)**

**d** Determine which project should be undertaken using expected utility as the criterion. **(1)**

**7** A company builds trailers and has overhead costs amounting to £10 000 per month for building up to 14 trailers. More than 14 trailers may be built in a month, for an additional overhead cost of £800.

The maximum number of trailers that can be built in any month is 17.

Up to 2 trailers can be stored in any month for a cost of £150 each.

Trailers are delivered at the end of each month. There are no trailers in stock at the beginning of July and there must be none left over at the end of October.

The order book for trailers is:

| **Month** | July | August | September | October |
|---|---|---|---|---|
| **Number ordered** | 15 | 13 | 17 | 15 |

Use dynamic programming to produce a schedule that minimises the production costs. **(12)**

# Answers

## CHAPTER 1

### Prior knowledge check

Let the number of large, medium and small paddling pools produced be $x$, $y$ and $z$ respectively.

Objective: Maximise $P = 5x + 4y + 2z$

subject to: $15x + 10y + 6z \leqslant 10000$

$x \geqslant 300$

$z < \frac{3}{7}(x + y)$

### Exercise 1A

**1 a**

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *A* | 28 | 4 | | 32 |
| *B* | | 41 | 3 | 44 |
| *C* | | | 34 | 34 |
| Demand | 28 | 45 | 37 | 110 |

**b** Supply points = 3, demand points = 3, occupied cells = 5.

$3 + 3 - 1 = 5$ = number of occupied cells.

**c** 22 434

**2 a**

| | *P* | *Q* | *R* | *S* | Supply |
|---|---|---|---|---|---|
| *A* | 21 | 32 | 1 | | 54 |
| *B* | | | 50 | 17 | 67 |
| *C* | | | | 29 | 29 |
| Demand | 21 | 32 | 51 | 46 | 150 |

**b** Supply points = 3, demand points = 4, occupied cells = 6.

$3 + 4 - 1 = 6$ = number of occupied cells.

**c** 5032

**3 a**

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *A* | 123 | | | 123 |
| *B* | 77 | 66 | | 143 |
| *C* | | 34 | 50 | 84 |
| *D* | | | 150 | 150 |
| Demand | 200 | 100 | 200 | 500 |

**b** Supply points = 4, demand points = 3, occupied cells = 6.

$4 + 3 - 1 = 6$ = number of occupied cells.

**c** 8680

**4 a**

| | *P* | *Q* | *R* | *S* | Supply |
|---|---|---|---|---|---|
| *A* | 134 | | | | 134 |
| *B* | 41 | 162 | | | 203 |
| *C* | | 13 | 163 | | 176 |
| *D* | | | 12 | 175 | 187 |
| Demand | 175 | 175 | 175 | 175 | 700 |

**b** Supply points = 4, demand points = 4, occupied cells = 7.

$4 + 4 - 1 = 7$ = number of occupied cells.

**c** 45 761

### Exercise 1B

**1 a** The total supply is 200, but the total demand is 180. A dummy is needed to absorb this excess, so that total supply equals total demand.

**b**

| | *A* | *B* | *C* | *D* | Dummy | Supply |
|---|---|---|---|---|---|---|
| *X* | 40 | 20 | | | | 60 |
| *Y* | | 50 | 10 | | | 60 |
| *Z* | | | 40 | 20 | 20 | 80 |
| Demand | 40 | 70 | 50 | 20 | 20 | 200 |

£53.80

**2 a** A degenerate solution occurs when the number of cells used in a solution is fewer than the number of rows + number of columns − 1. It will happen when an entry, other than the last, completes both the supply requirement of the row and the demand requirement of the column.

**b**

| | *K* | *L* | *M* | *N* | Supply |
|---|---|---|---|---|---|
| *A* | 20 | | | | 20 |
| *B* | 5 | 10 | | | 15 |
| *C* | | | 18 | 2 | 20 |
| *D* | | | | 20 | 20 |
| Demand | 25 | 10 | 18 | 22 | 75 |

Number of rows + number of columns − 1 = 7, but there are only 6 cells filled.

**c** A zero must be placed in one of the empty cells so that 6 cells are filled.

| | *K* | *L* | *M* | *N* | Supply |
|---|---|---|---|---|---|
| *A* | 20 | | | | 20 |
| *B* | 5 | 10 | | | 15 |
| *C* | | 0 | 18 | 2 | 20 |
| *D* | | | | 20 | 20 |
| Demand | 25 | 10 | 18 | 22 | 75 |

or

| | *K* | *L* | *M* | *N* | Supply |
|---|---|---|---|---|---|
| *A* | 20 | | | | 20 |
| *B* | 5 | 10 | 0 | | 15 |
| *C* | | | 18 | 2 | 20 |
| *D* | | | | 20 | 20 |
| Demand | 25 | 10 | 18 | 22 | 75 |

3 a $a = 10$ and $b = 9$

b

| | *L* | *M* | *N* | Supply |
|---|---|---|---|---|
| *P* | 15 | 7 | | 22 |
| *Q* | | 10 | | 10 |
| *R* | | | 11 | 11 |
| *S* | | | 9 | 9 |
| Demand | 15 | 17 | 20 | 52 |

c Include a zero in one of the empty cells (*QN* or *RM*).

4 a Total supply = 28 + 26 + 31 = 85 tyres
Total demand = 24 + 30 + 45 = 99 tyres
Total supply < Total demand, so a dummy row is needed

b Initial degenerate solution:

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *A* | 24 | 4 | | 28 |
| *B* | | 26 | | 26 |
| *C* | | | 31 | 31 |
| *D* | | | 14 | 14 |
| Demand | 24 | 30 | 45 | |

To avoid a degenerate solution requires 3 + 4 − 1 cells filled, so place a zero in *BR* or *CQ*.

c The value is row *D*, 14, is the shortfall in the number of tyres required by *R*.

## Exercise 1C

1 a

| Shadow costs | | 150 | 213 | 227 |
|---|---|---|---|---|
| | | *P* | *Q* | *R* |
| 0 | *A* | 150 | 213 | 222 |
| −9 | *B* | 175 | 204 | 218 |
| 19 | *C* | 188 | 198 | 246 |

b Improvement indices for cells:
$BP = 175 + 9 - 150 = 34$
$CP = 188 - 19 - 150 = 19$
$CQ = 198 - 19 - 213 = -34$
$AR = 222 - 0 - 227 = -5$

c Entering cell is *CQ*, since it has the most negative improvement index.

2 a

| Shadow costs | | 27 | 33 | 34 | 27 |
|---|---|---|---|---|---|
| | | *P* | *Q* | *R* | *S* |
| 0 | *A* | 27 | 33 | 34 | 41 |
| 3 | *B* | 31 | 29 | 37 | 30 |
| 8 | *C* | 40 | 32 | 28 | 35 |

b Improvement indices for cells:
$BP = 31 - 3 - 27 = 1$
$CP = 40 - 8 - 27 = 5$
$BQ = 29 - 3 - 33 = -7$
$CQ = 32 - 8 - 33 = -9$
$CR = 28 - 8 - 34 = -14$
$AS = 41 - 0 - 27 = 14$

c Entering cell is *CR*, since it has the most negative improvement index.

3 a

| Shadow costs | | 17 | 23 | 19 |
|---|---|---|---|---|
| | | *P* | *Q* | *R* |
| 0 | *A* | 17 | 24 | 19 |
| −2 | *B* | 15 | 21 | 25 |
| −1 | *C* | 19 | 22 | 18 |
| −3 | *D* | 20 | 27 | 16 |

b Improvement indices for cells:
$CP = 19 + 1 - 17 = 3$
$DP = 20 + 3 - 17 = 6$
$AQ = 24 - 0 - 23 = 1$
$DQ = 27 + 3 - 23 = 7$
$AR = 19 - 0 - 19 = 0$
$BR = 25 + 2 - 19 = 8$

c There are no negative improvement indices, so the solution is optimal.

4 a

| Shadow costs | | 56 | 73 | 60 | 56 |
|---|---|---|---|---|---|
| | | *P* | *Q* | *R* | *S* |
| 0 | *A* | 56 | 86 | 80 | 61 |
| 3 | *B* | 59 | 76 | 78 | 65 |
| −3 | *C* | 62 | 70 | 57 | 67 |
| 15 | *D* | 60 | 68 | 75 | 71 |

b Improvement indices for cells:
$CP = 62 + 3 - 56 = 9$
$DP = 60 - 15 - 56 = -11$
$AQ = 86 - 0 - 73 = 13$
$DQ = 68 - 15 - 73 = -20$
$AR = 80 - 0 - 60 = 20$
$BR = 78 - 3 - 60 = 15$
$AS = 61 - 0 - 56 = 5$
$BS = 65 - 3 - 56 = 6$
$CS = 67 + 3 - 56 = 14$

c Entering cell is *DQ*, since it has the most negative improvement index.

5 a Total supply = 76 + 68 + 60 = 204
Total demand = 83 + 57 + 64 = 204
Total supply = Total demand, so the problem is balanced.

b

| | *X* | *Y* | *Z* | Supply |
|---|---|---|---|---|
| *A* | 76 | | | 76 |
| *B* | 7 | 57 | 4 | 68 |
| *C* | | | 60 | 60 |
| Demand | 83 | 57 | 64 | 204 |

c

| Shadow costs | | 39 | 46 | 52 |
|---|---|---|---|---|
| | | *X* | *Y* | *Z* |
| 0 | *A* | 39 | 54 | 47 |
| 9 | *B* | 48 | 55 | 61 |
| 6 | *C* | 52 | 44 | 58 |

Improvement indices for cells:
$AY = 8$; $AZ = -5$; $CX = 7$; $CY = -8$
There are negative improvement indices, so the solution may not be optimal.

## Exercise 1D

**1** Solution

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *A* | 28 | | 4 | 32 |
| *B* | | 11 | 33 | 44 |
| *C* | | 34 | | 34 |
| **Demand** | 28 | 45 | 37 | 110 |

Cost 21 258

Shadow costs and improvement indices:

| | | 150 | 208 | 222 |
|---|---|---|---|---|
| | | *P* | *Q* | *R* |
| 0 | *A* | × | 5 | × |
| −4 | *B* | 29 | × | × |
| −10 | *C* | 48 | × | 34 |

All improvement indices are non-negative, so the solution is optimal.

**2** Solution

| | *P* | *Q* | *R* | *S* | Supply |
|---|---|---|---|---|---|
| *A* | 21 | 11 | 22 | | 57 |
| *B* | | 21 | | 46 | 67 |
| *C* | | | 29 | | 29 |
| **Demand** | 21 | 32 | 51 | 46 | 150 |

Cost 4479

Shadow costs and improvement indices:

| | | 27 | 33 | 34 | 34 |
|---|---|---|---|---|---|
| | | *P* | *Q* | *R* | *S* |
| 0 | *A* | × | × | × | 7 |
| −4 | *B* | 8 | × | 7 | × |
| −6 | *C* | 19 | 5 | × | 7 |

All improvement indices are non-negative, so the solution is optimal.

**3** Solution

| | *P* | *Q* | *R* | *S* | Supply |
|---|---|---|---|---|---|
| *A* | | | | 134 | 134 |
| *B* | 163 | | | 40 | 203 |
| *C* | | | 175 | 1 | 176 |
| *D* | 12 | 175 | | | 187 |
| **Demand** | 175 | 175 | 175 | 175 | 700 |

Cost 43 053

Shadow costs and improvement indices:

| | | 55 | 63 | 51 | 61 |
|---|---|---|---|---|---|
| | | *P* | *Q* | *R* | *S* |
| 0 | *A* | 1 | 23 | 29 | × |
| 4 | *B* | × | 9 | 23 | × |
| 6 | *C* | 1 | 1 | × | × |
| 5 | *D* | × | × | 19 | 5 |

All improvement indices are non-negative, so the solution is optimal.

**4** Initial solution

| | *P* | *Q* | Supply |
|---|---|---|---|
| *A* | 3 | | 3 |
| *B* | 3 | 2 | 5 |
| *C* | | 2 | 2 |
| **Demand** | 6 | 4 | 10 |

Initial cost £44

**First iteration**

Shadow costs and improvement indices:

| | | 2 | 7 |
|---|---|---|---|
| | | *P* | *Q* |
| 0 | *A* | × | −1 |
| 0 | *B* | × | × |
| 2 | *C* | 2 | × |

Entering cell *AQ*.

| | *P* | *Q* |
|---|---|---|
| *A* | $3 - \theta$ | $\theta$ |
| *B* | $3 + \theta$ | $2 - \theta$ |
| *C* | | 2 |

$\theta = 2$, exiting cell *BQ*.

Improved solution

| | *P* | *Q* |
|---|---|---|
| *A* | 1 | 2 |
| *B* | 5 | |
| *C* | | 2 |

Improved cost £42

Shadow costs and improvement indices:

| | | 2 | 6 |
|---|---|---|---|
| | | *P* | *Q* |
| 0 | *A* | × | × |
| 0 | *B* | × | 1 |
| 3 | *C* | 1 | × |

All improvement indices are non-negative, so the solution is optimal.

**5** **a** Total supply = 17 + 14 + 14 = 45, total demand = 12 + 15 + 18 = 45

Total supply = total demand, so the problem is balanced.

**b**

| | *A* | *B* | *C* | Supply |
|---|---|---|---|---|
| *X* | 12 | 5 | | 17 |
| *Y* | | 10 | 4 | 14 |
| *Z* | | | 14 | 14 |
| **Demand** | 12 | 15 | 18 | 45 |

**c** Shadow costs and improvement indices:

| | | 16 | 21 | 18 |
|---|---|---|---|---|
| | | *A* | *B* | *C* |
| 0 | *X* | × | × | −3 |
| 1 | *Y* | 6 | × | × |
| −2 | *Z* | 4 | 5 | × |

*XC* is negative, so the solution may not be optimal.

**d** Entering cell $XC$.

| | A | B | C |
|---|---|---|---|
| X | 12 | $5 - \theta$ | $\theta$ |
| Y | | $10 + \theta$ | $4 - \theta$ |
| Z | | | 14 |

$\theta = 4$, exiting cell $YC$.
Improved solution

| | A | B | C | Supply |
|---|---|---|---|---|
| X | 12 | 1 | 4 | 17 |
| Y | | 14 | | 14 |
| Z | | | 14 | 14 |
| **Demand** | 12 | 15 | 18 | 45 |

**e** Shadow costs and improvement indices:

| | | 16 | 21 | 15 |
|---|---|---|---|---|
| | | A | B | C |
| 0 | X | × | × | × |
| 1 | Y | 6 | × | 3 |
| 1 | Z | 1 | 2 | × |

All improvement indices are non-negative, so the solution is optimal.
Cost £805

**6 a** Total supply = 73 units and Total demand = 56 units. A dummy demand point is needed as supply is greater than demand.

**b**

| | D | E | F | G | Supply |
|---|---|---|---|---|---|
| A | 22 | 4 | | | 26 |
| B | | 12 | 11 | | 23 |
| C | | | 7 | 17 | 24 |
| **Demand** | 22 | 16 | 18 | 17 | 73 |

**c** Shadow costs and improvement indices:

| | | 24 | 32 | 39 | 9 |
|---|---|---|---|---|---|
| | | D | E | F | G |
| 0 | A | × | × | −14 | −9 |
| −11 | B | 14 | × | × | 2 |
| −9 | C | 4 | 3 | × | × |

**d** Entering cell $AF$.

| | D | E | F | G | Supply |
|---|---|---|---|---|---|
| A | 22 | $4 - \theta$ | $\theta$ | | 26 |
| B | | $12 + \theta$ | $11 - \theta$ | | 23 |
| C | | | 7 | 17 | 24 |
| **Demand** | 22 | 16 | 18 | 17 | 73 |

$\theta = 4$, exiting cell $AE$.
Improved solution

| | D | E | F | G | Supply |
|---|---|---|---|---|---|
| A | 22 | | 4 | | 26 |
| B | | 16 | 7 | | 23 |
| C | | | 7 | 17 | 24 |
| **Demand** | 22 | 16 | 18 | 17 | 73 |

**7 a** Total supply = 99 units
Total demand = 114 units
114 − 90 = 15
A dummy supply point is needed providing 15 units at zero cost.

**b** Initial solution:

| | X | Y | Z | Supply |
|---|---|---|---|---|
| A | 30 | | | 30 |
| B | 5 | 29 | | 34 |
| C | | 8 | 27 | 35 |
| D | | | 15 | 15 |
| **Demand** | 35 | 37 | 42 | |

Shadow costs and improvement indices:

| | | 18 | 11 | 7 |
|---|---|---|---|---|
| | | X | Y | Z |
| 0 | A | × | 14 | 14 |
| 8 | B | × | × | 12 |
| 17 | C | −15 | × | × |
| −7 | D | −11 | −4 | × |

Entering cell $CX$; $\theta = 5$; Exiting cell $BX$

| | X | Y | Z |
|---|---|---|---|
| A | 30 | | |
| B | $5 - \theta$ | $29 + \theta$ | |
| C | $\theta$ | $8 - \theta$ | 27 |
| D | | | 15 |

Improved solution:

| | X | Y | Z |
|---|---|---|---|
| A | 30 | | |
| B | | 34 | |
| C | 5 | 3 | 27 |
| D | | | 15 |

Shadow costs and improvement indices:

| | | 18 | 26 | 22 |
|---|---|---|---|---|
| | | X | Y | Z |
| 0 | A | × | −1 | −1 |
| −7 | B | 15 | × | 12 |
| 2 | C | × | × | × |
| −22 | D | 4 | −4 | × |

Entering cell $DY$; $\theta = 3$; Exiting cell $CY$

| | X | Y | Z |
|---|---|---|---|
| A | 30 | | |
| B | | 34 | |
| C | 5 | $3 - \theta$ | $27 + \theta$ |
| D | | $\theta$ | $15 - \theta$ |

Improved solution:

| | X | Y | Z |
|---|---|---|---|
| A | 30 | | |
| B | | 34 | |
| C | 5 | | 30 |
| D | | 3 | 12 |

Shadow costs and improvement indices:

| | | 18 | 22 | 22 |
|---|---|---|---|---|
| | | **X** | **Y** | **Z** |
| 0 | **A** | × | 3 | −1 |
| −3 | **B** | 11 | × | 8 |
| 2 | **C** | × | 6 | × |
| −22 | **D** | 4 | × | × |

Entering cell *AZ*; $\theta = 30$; Exiting cell *AX*
In this case, we could have used either *AX* or *CZ* as the exiting cell.

| | **X** | **Y** | **Z** |
|---|---|---|---|
| **A** | $30 - \theta$ | | $\theta$ |
| **B** | | 34 | |
| **C** | $5 + \theta$ | | $30 - \theta$ |
| **D** | | 3 | 12 |

Improved solution:

| | **X** | **Y** | **Z** |
|---|---|---|---|
| **A** | | | 30 |
| **B** | | 34 | |
| **C** | 35 | | 0 |
| **D** | | 3 | 12 |

Shadow costs and improvement indices:

| | | 17 | 21 | 21 |
|---|---|---|---|---|
| | | **X** | **Y** | **Z** |
| 0 | **A** | 1 | 4 | × |
| −2 | **B** | 11 | × | 8 |
| 3 | **C** | × | 4 | × |
| −21 | **D** | 4 | × | × |

All the improvement indices are non-negative so the solution is optimal.
Total cost = 1976

## Exercise 1E

**1** Let $x_{ij}$ be the number of units transported from $i$ to $j$ where
$i \in \{A, B, C\}$
$j \in \{P, Q, R\}$
$x_{ij} \geqslant 0$
Minimise $C = 150x_{AP} + 213x_{AQ} + 222x_{AR} + 175x_{BP} + 204x_{BQ} + 218x_{BR} + 188x_{CP} + 198x_{CQ} + 246x_{CR}$
subject to $x_{AP} + x_{AQ} + x_{AR} \leqslant 32$
$x_{BP} + x_{BQ} + x_{BR} \leqslant 44$
$x_{CP} + x_{CQ} + x_{CR} \leqslant 34$
$x_{AP} + x_{BP} + x_{CP} \geqslant 28$
$x_{AQ} + x_{BQ} + x_{CQ} \geqslant 45$
$x_{AR} + x_{BR} + x_{CR} \geqslant 37$

**2** Let $x_{ij}$ be the number of units transported from $i$ to $j$ where
$i \in \{A, B, C\}$
$j \in \{P, Q, R, S\}$
$x_{ij} \geqslant 0$
Minimise $C = 27x_{AP} + 33x_{AQ} + 34x_{AR} + 41x_{AS} + 31x_{BP} + 29x_{BQ} + 37x_{BR} + 30x_{BS} + 40x_{CP} + 32x_{CQ} + 28x_{CR} + 35x_{CS}$
subject to $\Sigma x_{Aj} \leqslant 54$ $\quad \Sigma x_{iP} \geqslant 21$
$\Sigma x_{Bj} \leqslant 67$ $\quad \Sigma x_{iQ} \geqslant 32$
$\Sigma x_{Cj} \leqslant 29$ $\quad \Sigma x_{iR} \geqslant 51$
$\Sigma x_{iS} \geqslant 46$

**3** Let $x_{ij}$ be the number of units transported from $i$ to $j$ where
$i \in \{A, B, C, D\}$
$j \in \{P, Q, R\}$
$x_{ij} \geqslant 0$
Minimise $C = 17x_{AP} + 24x_{AQ} + 19x_{AR} + 15x_{BP} + 21x_{BQ} + 25x_{BR} + 19x_{CP} + 22x_{CQ} + 18x_{CR} + 20x_{DP} + 27x_{DQ} + 16x_{DR}$
subject to $\Sigma x_{Aj} \leqslant 123$ $\quad \Sigma x_{iP} \geqslant 200$
$\Sigma x_{Bj} \leqslant 143$ $\quad \Sigma x_{iQ} \geqslant 100$
$\Sigma x_{Cj} \leqslant 84$ $\quad \Sigma x_{iR} \geqslant 200$
$\Sigma x_{Dj} \leqslant 150$

**4** Let $x_{ij}$ be the number of units transported from $i$ to $j$ where
$i \in \{A, B, C, D\}$
$j \in \{P, Q, R, S\}$
$x_{ij} \geqslant 0$
Minimise $C = 56x_{AP} + 86x_{AQ} + 80x_{AR} + 61x_{AS} + 59x_{BP} + 76x_{BQ} + 78x_{BR} + 65x_{BS} + 62x_{CP} + 70x_{CQ} + 57x_{CR} + 67x_{CS} + 60x_{DP} + 68x_{DQ} + 75x_{DR} + 71x_{DS}$
subject to $\Sigma x_{Aj} \leqslant 134$ $\quad \Sigma x_{iP} \geqslant 175$
$\Sigma x_{Bj} \leqslant 203$ $\quad \Sigma x_{iQ} \geqslant 175$
$\Sigma x_{Cj} \leqslant 176$ $\quad \Sigma x_{iR} \geqslant 175$
$\Sigma x_{Dj} \leqslant 187$ $\quad \Sigma x_{iS} \geqslant 175$

**5** **a** Total supply = 25 + 28 + 21 = 74
Total demand = 20 + 15 + 12 + 16 = 63
Total supply ≠ Total demand, so the problem is unbalanced.

**b** Let $x_{ij}$ represent the number of televisions transported from $i$ to $j$ where $i \in \{A, B, C, D\}$ and $j \in \{W, X, Y, Z\}$ and $x_{ij} \geqslant 0$
Minimise $C = 8x_{AW} + 11x_{AX} + 7x_{AY} + 9x_{AZ} + 12x_{BW} + 10x_{BX} + 8x_{BY} + 7x_{BZ} + 10x_{CW} + 12x_{CX} + 9x_{CY} + 8x_{CZ}$
subject to $\Sigma x_{Aj} \leqslant 25$ $\quad \Sigma x_{iW} \geqslant 20$
$\Sigma x_{Bj} \leqslant 28$ $\quad \Sigma x_{iX} \geqslant 15$
$\Sigma x_{Cj} \leqslant 21$ $\quad \Sigma x_{iY} \geqslant 12$
$\Sigma x_{Dj} \geqslant 11$ $\quad \Sigma x_{iZ} \geqslant 16$

**6** **a** The decision variables have not been defined. The objective function is to be minimised. The third constraint should be ⩽ 10 not ⩽ 20. The last three constraints should all be ⩾ not ⩽.

**b** Let $x_{ij}$ represent the number of cars transported from $i$ to $j$ where
$i \in \{A, B, C\}$
$j \in \{X, Y, Z\}$
$x_{ij} \geqslant 1$
Minimise $C = 70x_{AX} + 50x_{AY} + 60x_{AZ} + 85x_{BX} + 60x_{BY} + 74x_{BZ} + 68x_{CX} + 73x_{CY} + 80x_{CZ}$
subject to $x_{AX} + x_{AY} + x_{AZ} \leqslant 12$
$x_{BX} + x_{BY} + x_{BZ} \leqslant 8$
$x_{CX} + x_{CY} + x_{CZ} \leqslant 10$
$x_{AX} + x_{BX} + x_{CX} \geqslant 11$
$x_{AY} + x_{BY} + x_{CY} \geqslant 9$
$x_{AZ} + x_{BZ} + x_{CZ} \geqslant 6$

## Mixed exercise 1

**1 a**

| | *L* | *M* | Supply |
|---|---|---|---|
| *A* | 15 | | 15 |
| *B* | 1 | 4 | 5 |
| *C* | | 8 | 8 |
| **Demand** | 16 | 12 | 28 |

**b** Shadow costs and improvement indices:

| | | 20 | 10 |
|---|---|---|---|
| | | *L* | *M* |
| 0 | *A* | × | 60 |
| 20 | *B* | × | × |
| 80 | *C* | −40 | × |

Entering cell *CL*.

| | *L* | *M* | Supply |
|---|---|---|---|
| *A* | 15 | | 15 |
| *B* | $1-\theta$ | $4+\theta$ | 5 |
| *C* | $\theta$ | $8-\theta$ | 8 |
| **Demand** | 16 | 12 | 28 |

$\theta = 1$, exiting cell *BL*.
Improved solution

| | *L* | *M* | Supply |
|---|---|---|---|
| *A* | 15 | | 15 |
| *B* | | 5 | 5 |
| *C* | 1 | 7 | 8 |
| **Demand** | 16 | 12 | 28 |

**c** Shadow costs and improvement indices:

| | | 20 | 50 |
|---|---|---|---|
| | | *L* | *M* |
| 0 | *A* | × | 20 |
| −20 | *B* | 40 | × |
| 40 | *C* | × | × |

All improvement indices are non-negative, so the solution is optimal.
Cost £1140

**d** Let $x_{ij}$ be the number of units transported from $i$ to $j$ where
$i \in \{A, B, C\}$
$j \in \{L, M\}$
$x_{ij} \geqslant 0$
Minimise $C = 20x_{AL} + 70x_{AM} + 40x_{BL} + 30x_{BM} + 60x_{CL} + 90x_{CM}$
subject to $\Sigma x_{Aj} \leqslant 15$ $\quad \Sigma x_{iA} \geqslant 16$
$\Sigma x_{Bj} \leqslant 5$ $\quad \Sigma x_{iB} \geqslant 12$
$\Sigma x_{Cj} \leqslant 8$

**2 a**

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *F* | 10 | 5 | | 15 |
| *G* | | 25 | 10 | 35 |
| *H* | | | 10 | 10 |
| **Demand** | 10 | 30 | 20 | 60 |

**b First iteration**
Shadow costs and improvement indices:

| | | 23 | 21 | 22 |
|---|---|---|---|---|
| | | *P* | *Q* | *R* |
| 0 | *F* | × | × | 0 |
| 2 | *G* | −4 | × | × |
| 1 | *H* | −2 | −1 | × |

Entering cell *GP*.

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *F* | $10-\theta$ | $5+\theta$ | | 15 |
| *G* | $\theta$ | $25-\theta$ | 10 | 35 |
| *H* | | | 10 | 10 |
| **Demand** | 10 | 30 | 20 | 60 |

$\theta = 10$, exiting cell *FP*.
Improved solution

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *F* | | 15 | | 15 |
| *G* | 10 | 15 | 10 | 35 |
| *H* | | | 10 | 10 |
| **Demand** | 10 | 30 | 20 | 60 |

**Second iteration**
Shadow costs and improvement indices:

| | | 19 | 21 | 22 |
|---|---|---|---|---|
| | | *P* | *Q* | *R* |
| 0 | *F* | 4 | × | 0 |
| 2 | *G* | × | × | × |
| 1 | *H* | 2 | −1 | × |

Entering cell *HQ*.

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *F* | | 15 | | 15 |
| *G* | 10 | $15-\theta$ | $10+\theta$ | 35 |
| *H* | | $\theta$ | $10-\theta$ | 10 |
| **Demand** | 10 | 30 | 20 | 60 |

$\theta = 10$, exiting cell *HR*.
Improved solution

| | *P* | *Q* | *R* | Supply |
|---|---|---|---|---|
| *F* | | 15 | | 15 |
| *G* | 10 | 5 | 20 | 35 |
| *H* | | 10 | | 10 |
| **Demand** | 10 | 30 | 20 | 60 |

**c** Shadow costs and improvement indices for solution:

| | | 19 | 21 | 22 |
|---|---|---|---|---|
| | | *P* | *Q* | *R* |
| 0 | *F* | 4 | × | 0 |
| 2 | *G* | × | × | × |
| 0 | *H* | 3 | × | 1 |

All improvement indices are non-negative, so the solution is optimal.

**d** 1330

**e**

| | *P* | *Q* | *R* | **Supply** |
|---|---|---|---|---|
| *F* | | | 15 | 15 |
| *G* | 10 | 20 | 5 | 35 |
| *H* | | 10 | | 10 |
| **Demand** | 10 | 30 | 20 | 60 |

Cost 1330

**3 a** Otherwise solution would be degenerate

**b** £1675

**c** Entering cell *MX*.

| | *X* | *Y* | *Z* | **Supply** |
|---|---|---|---|---|
| *J* | $25 - \theta$ | $5 + \theta$ | | 30 |
| *K* | | 40 | | 40 |
| *L* | | $0 - \theta$ | $50 + \theta$ | 50 |
| *M* | $\theta$ | | $50 - \theta$ | 50 |
| **Demand** | 25 | 45 | 100 | 170 |

$\theta = 0$, exiting cell *LY*.
Improved solution

| | *X* | *Y* | *Z* | **Supply** |
|---|---|---|---|---|
| *J* | 25 | 5 | | 30 |
| *K* | | 40 | | 40 |
| *L* | | | 50 | 50 |
| *M* | 0 | | 50 | 50 |
| **Demand** | 25 | 45 | 100 | 170 |

Cost £1675

**d** Not optimal since there are negative improvement indices.

**e** Shadow costs and improvement indices:

| | | 6 | 3 | 7 |
|---|---|---|---|---|
| | | *X* | *Y* | *Z* |
| 0 | *J* | 2 | 2 | × |
| 2 | *K* | −3 | × | × |
| 3 | *L* | −2 | −4 | × |
| 0 | *M* | × | × | 8 |

Entering cell *LY*.

| | *X* | *Y* | *Z* | **Supply** |
|---|---|---|---|---|
| *J* | | | 30 | 30 |
| *K* | | $20 - \theta$ | $20 + \theta$ | 40 |
| *L* | | $\theta$ | $50 - \theta$ | 50 |
| *M* | 25 | 25 | | 50 |
| **Demand** | 25 | 45 | 100 | 170 |

$\theta = 20$, exiting cell *KY*.
Improved solution

| | *X* | *Y* | *Z* | **Supply** |
|---|---|---|---|---|
| *J* | | | 30 | 30 |
| *K* | | | 40 | 40 |
| *L* | | 20 | 30 | 50 |
| *M* | 25 | 25 | | 50 |
| **Demand** | 25 | 45 | 100 | 170 |

Cost £1135

**f** It does not affect the total costs since it is not part of the optimal route.

**4 a** Dummy demand needed to absorb surplus stock

**b**

| | *S* | *T* | *U* | **Dummy** | **Stock** |
|---|---|---|---|---|---|
| *A* | 50 | | | | 50 |
| *B* | 50 | 20 | | | 70 |
| *C* | | 10 | 20 | 20 | 50 |
| **Demand** | 100 | 30 | 20 | 20 | 170 |

**c** **First iteration**

Shadow costs and improvement indices, using *V* to represent the dummy:

| | | 6 | 4 | 4 | −3 |
|---|---|---|---|---|---|
| | | *S* | *T* | *U* | *V* |
| 0 | *A* | × | 6 | 3 | 3 |
| 1 | *B* | × | × | 3 | 2 |
| 3 | *C* | −3 | × | × | × |

Entering cell *CS*.

| | *S* | *T* | *U* | *V* | **Supply** |
|---|---|---|---|---|---|
| *A* | 50 | | | | 50 |
| *B* | $50 - \theta$ | $20 + \theta$ | | | 70 |
| *C* | $\theta$ | $10 - \theta$ | 20 | 20 | 50 |
| **Demand** | 100 | 30 | 20 | 20 | 170 |

$\theta = 10$, exiting cell *CT*.
Improved solution

| | *S* | *T* | *U* | *V* | **Supply** |
|---|---|---|---|---|---|
| *A* | 50 | | | | 50 |
| *B* | 40 | 30 | | | 70 |
| *C* | 10 | | 20 | 20 | 50 |
| **Demand** | 100 | 30 | 20 | 20 | 170 |

**Second iteration**

Shadow costs and improvement indices:

| | | 6 | 4 | 7 | 0 |
|---|---|---|---|---|---|
| | | *S* | *T* | *U* | *V* |
| 0 | *A* | × | 6 | 0 | 0 |
| 1 | *B* | × | × | 0 | −1 |
| 0 | *C* | × | 3 | × | × |

Entering cell *BV*.

| | *S* | *T* | *U* | *V* | **Supply** |
|---|---|---|---|---|---|
| *A* | 50 | | | | 50 |
| *B* | $40 - \theta$ | 30 | | $\theta$ | 70 |
| *C* | $10 + \theta$ | | 20 | $20 - \theta$ | 50 |
| **Demand** | 100 | 30 | 20 | 20 | 170 |

$\theta = 20$, exiting cell *CV*.
Improved solution

| | *S* | *T* | *U* | *V* | **Supply** |
|---|---|---|---|---|---|
| *A* | 50 | | | | 50 |
| *B* | 20 | 30 | | 20 | 70 |
| *C* | 30 | | 20 | | 50 |
| **Demand** | 100 | 30 | 20 | 20 | 170 |

Shadow costs and improvement indices:

| | | 6 | 4 | 7 | −1 |
|---|---|---|---|---|---|
| | | **S** | **T** | **U** | **V** |
| 0 | **A** | × | 6 | 0 | 1 |
| 1 | **B** | × | × | 0 | × |
| 0 | **C** | × | 3 | × | 1 |

All improvement indices are non-negative, so the solution is optimal.

**d** £910

**e** Let $x_{ij}$ represent the cost of transporting one van load of fruit tree seedlings from $i$ to $j$ where $i \in \{A, B, C\}$ and $j \in \{S, T, U, V\}$ (using $V$ for the dummy) and $x_{ij} \geqslant 0$

Minimise $C = 6x_{AS} + 10x_{AT} + 7x_{AU} + 7x_{BS} + 5x_{BT} + 8x_{BU} + 6x_{CS} + 7x_{CT} + 7x_{CU}$

subject to $\Sigma x_{Aj} \leqslant 50$ $\quad \Sigma x_{iS} \geqslant 100$
$\Sigma x_{Bj} \leqslant 70$ $\quad \Sigma x_{iT} \geqslant 30$
$\Sigma x_{Cj} \leqslant 50$ $\quad \Sigma x_{iU} \geqslant 20$

**5** Let $x_{ij}$ represent the cost of transporting one roll of carpet from $i$ to $j$ where $i \in \{A, B, C\}$ and $j \in \{P, Q, R, S\}$ and $x_{ij} \geqslant 0$

Minimise $C = 28x_{AP} + 12x_{AQ} + 19x_{AR} + 16x_{AS} + 31x_{BP} + 28x_{BQ} + 23x_{BR} + 19x_{BS} + 18x_{CP} + 21x_{CQ} + 22x_{CR} + 28x_{CS}$

subject to $\Sigma x_{Aj} \leqslant 28$ $\quad \Sigma x_{iP} \geqslant 16$
$\Sigma x_{Bj} \leqslant 33$ $\quad \Sigma x_{iQ} \geqslant 20$
$\Sigma x_{Cj} \leqslant 18$ $\quad \Sigma x_{iR} \geqslant 26$
$\Sigma x_{iS} \geqslant 17$

**Challenge**

**a**

| | **P** | **Q** | **R** | **S** | **Stock** |
|---|---|---|---|---|---|
| **A** | 7 | 8 | 6 | 3 | 14 |
| **B** | 5 | 7 | 9 | 5 | 12 |
| **C** | 6 | 8 | 8 | 4 | 16 |
| **Demand** | 15 | 9 | 11 | 7 | |

**b** Let $x_{ij}$ represent the cost of transporting one unit of stock from $i$ to $j$ where $i \in \{A, B, C\}$ and $j \in \{P, Q, R\}$
Let $x_{iS}$ represent the cost of storing 1 surplus unit at warehouse $i$ for a dummy retailer $S$.

Minimise $C = 7x_{AP} + 8x_{AQ} + 6x_{AR} + 3x_{AS} + 5x_{BP} + 7x_{BQ} + 9x_{BR} + 5x_{BS} + 6x_{CP} + 8x_{CQ} + 8x_{CR} + 4x_{CS}$

subject to $\Sigma x_{Aj} \leqslant 14$ $\quad \Sigma x_{iP} \geqslant 15$
$\Sigma x_{Bj} \leqslant 12$ $\quad \Sigma x_{iQ} \geqslant 9$
$\Sigma x_{Cj} \leqslant 16$ $\quad \Sigma x_{iR} \geqslant 11$
$\Sigma x_{iS} \geqslant 7$

## CHAPTER 2

### Prior knowledge check

Let $x_A$ be the number of type $A$ saws produced each week and let $x_B$ be the number of type $B$ saws produced each week.
Maximise $80x_A + 60x_B$
subject to
$x_A \geqslant 40$
$5x_A + 2x_B \leqslant 150$
$x_A + x_B < 10x_A < 3(x_A + x_B)$

### Exercise 2A

**1** Two solutions

$A$–$Y$ (35), $B$–$Z$ (27), $C$–$X$ (30) *or* $A$–$Z$ (31), $B$–$Y$ (31), $C$–$X$ (30)

Cost £92

**2** Three solutions

$P$–$A$ (34), $Q$–$B$ (32), $R$–$D$ (36), $S$–$C$ (35) *or* $P$–$D$ (32), $Q$–$A$ (35), $R$–$B$ (35), $S$–$C$ (35) *or* $P$–$C$ (32), $Q$–$A$ (35), $R$–$D$ (36), $S$–$B$ (34)

Cost £137

**3** $J$–$R$ (20)
$K$–$U$ (20)
$L$–$S$ (10)
$M$–$T$ (9)
Cost £59

**4** Two solutions

$D$–$Z$ (80), $E$–$X$ (95), $F$–$V$ (90), $G$–$Y$ (85), $H$–$W$ (100) *or* $D$–$Y$ (87), $E$–$X$ (95), $F$–$V$ (90), $G$–$W$ (83), $H$–$Z$ (95)

Cost £450

**5 a**

| | **100 m** | **Hurdles** | **200 m** | **400 m** |
|---|---|---|---|---|
| **A** | 0 | 0 | 0 | 0 |
| **B** | 0 | 2 | 4 | 5 |
| **C** | 0 | 1 | 3 | 8 |
| **D** | 0 | 1 | 3 | 11 |

**b** Two solutions

$A$–400 m (64), $B$–100 m (13), $C$–Hurdles (20), $D$–200 m (39) *or* $A$–400 m (64), $B$–100 m (13), $C$–200 m (38), $D$–Hurdles (21)

Time 136 seconds

**c** 4 lines are needed to cover the zeros.

**6 a** $A$–Eucalyptus £62
$B$–Willow £72
$C$–Elm £84
$D$–Beech £145
$E$–Oak £138
$F$–Olive £80

**b** Minimum cost = £581

### Exercise 2B

**1** $J$–$M$ (23)
$K$–dummy
$L$–$N$ (28)
Cost £51

**2** $A$–$Z$ (35)
$B$–$Y$ (10)
$C$–$W$ (24)
dummy–$X$ (0)
Cost £69

**3** Two solutions

$W$–dummy, $X$–$T$ (48), $Y$–$S$ (38), $Z$–$R$ (73) *or* $W$–$T$ (55), $X$–$R$ (67), $Y$–dummy, $Z$–$S$ (37)

Cost £159

**4** $P$ – dummy
$Q$ – $F$ (39)
$R$ – $G$ (22)
$S$ – $H$ (29)
$T$ – $E$ (18)
Cost £108

**5 a** There are more tasks than workers. An extra worker is required to make the problem balanced and this is shown as a dummy row.

**b** $A$ – 5, $B$ – 1, $C$ – 3, $D$ – 6, $E$ – 4 or $A$ – 2, $B$ – 1, $C$ – 3, $D$ – 6, $E$ – 4, the minimum cost is £294.

**c** Depending on answer to part **b**, task 2 or task 5 will not be completed as it is assigned to a dummy worker.

## Exercise 2C

**1** $L$ – $C$ (37)
$M$ – $E$ (16)
$N$ – $D$ (41)
Profit £94

**2** Two solutions

| | | |
|---|---|---|
| $C$ – $T$ (34) | | $C$ – $V$ (35) |
| $D$ – $U$ (34) | *or* | $D$ – $U$ (34) |
| $E$ – $S$ (42) | | $E$ – $S$ (42) |
| $F$ – $V$ (35) | | $F$ – $T$ (34) |

Profit £145

**3** $R$ – $F$ (22)
$S$ – $E$ (20)
$T$ – $G$ (18)
$U$ – $H$ (28)
Profit £88

**4 a** To find a maximum choose the largest value in the cost matrix and subtract every entry from that value.

**b** $A$ – Research (95)
$B$ – Farm (110)
$C$ – Explore (105)
$D$ – Build (84)
$E$ – Mine (120)
Maximum 514 points

**5 a** $A$ – $Z$, $B$ – $Y$, $C$ – $X$, $D$ – $V$, $E$ – $W$

**b** Maximum profit = £477

## Exercise 2D

**1** $P$ – $L$ (48)
$Q$ – $M$ (37)
$R$ – $N$ (56)
Cost £141

**2** $R$ – $E$ (47)
$S$ – $D$ (32)
$T$ – $F$ (43)
$U$ – $G$ (47)
Cost £169

**3** $A$ – $Q$ (53)
$B$ – $R$ (61)
$C$ – $S$ (62)
$D$ – $P$ (39)
Cost £215

**4 a** Enter a large value into the matrix to make those assignments 'unattractive'.

**b**

| | | |
|---|---|---|
| $J$ – $R$ (143) | | $J$ – $T$ |
| $K$ – $S$ (106) | | $K$ – $S$ |
| $L$ – $T$ (143) | or | $L$ – $R$ |
| $M$ – $U$ (134) | | $M$ – $U$ |
| $N$ – $V$ (253) | | $N$ – $V$ |

Time 779 minutes

**5 a** $P$ – 4, $Q$ – 1, $R$ – 3, $S$ – 5, $T$ – 2

**b** Maximum profit = £237

**Challenge**

**a** Reduce the cost to 0 for that cell to make it 'attractive' or delete row for worker 2 and column for machine $D$.

**b** Assign large value to cell for Worker 2/Machine $E$ so as to make it 'unattractive'

**c** Machine $D$ should have two workers trained as all column $D$'s elements are smaller than all the other elements in the table. To allocate two workers, replicate Machine $D$ as Machine $E$.

## Exercise 2E

**1** Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$

where $i \in \{L, M, N\}$
$j \in \{C, D, E\}$

Minimise $C = 37x_{LC} + 15x_{LD} + 12x_{LE} + 25x_{MC} + 13x_{MD} + 16x_{ME} + 32x_{NC} + 41x_{ND} + 35x_{NE}$

subject to $\Sigma x_{Lj} = 1$ $\quad \Sigma x_{iC} = 1$
$\Sigma x_{Mj} = 1$ $\quad \Sigma x_{iD} = 1$
$\Sigma x_{Nj} = 1$ $\quad \Sigma x_{iE} = 1$

**2** Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$

where $i \in \{C, D, E, F\}$
$j \in \{S, T, U, V\}$

Minimise $C = 36x_{CS} + 34x_{CT} + 32x_{CU} + 35x_{CV} + 37x_{DS} + 32x_{DT} + 34x_{DU} + 33x_{DV} + 42x_{ES} + 35x_{ET} + 37x_{EU} + 36x_{EV} + 39x_{FS} + 34x_{FT} + 35x_{FU} + 35x_{FV}$

subject to $\Sigma x_{Cj} = 1$ $\quad \Sigma x_{iS} = 1$
$\Sigma x_{Dj} = 1$ $\quad \Sigma x_{iT} = 1$
$\Sigma x_{Ej} = 1$ $\quad \Sigma x_{iU} = 1$
$\Sigma x_{Fj} = 1$ $\quad \Sigma x_{iV} = 1$

**3** Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$

where $i \in \{A, B, C, D\}$
$j \in \{1, 2, 3, 4\}$

Maximise $P = 11x_{A1} + 15x_{A2} + 50x_{A3} + 14x_{B1} + 18x_{B2} + 17x_{B3} + 16x_{C1} + 13x_{C2} + 23x_{C3} + 15x_{D1} + 14x_{D2} + 22x_{D2}$

subject to $\Sigma x_{Aj} = 1$ $\quad \Sigma x_{i1} = 1$
$\Sigma x_{Bj} = 1$ $\quad \Sigma x_{i2} = 1$
$\Sigma x_{Cj} = 1$ $\quad \Sigma x_{i3} = 1$
$\Sigma x_{Dj} = 1$ $\quad \Sigma x_{i4} = 1$

**4 a** Subtract the value in each cell from the maximum value in the matrix.

**b** $A$ – $W$, $B$ – $Z$, $C$ – $Y$, $D$ – $X$, Profit 48

**c** Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$

where $x_{ij}$ represents worker $i$ being assigned to task $j$
$i \in \{A, B, C, D\}$
$j \in \{W, X, Y, Z\}$

Minimise $P = 2x_{AW} + 6x_{AX} + 3x_{AY} + 5x_{AZ} + 4x_{BX} + 5x_{BY} + x_{BZ} + 3x_{CW} + 5x_{CX} + 2x_{CY} + 4x_{CZ} + x_{DW} + 3x_{DX} + 4x_{DY} + 2x_{DZ}$

or Maximise $P = 12x_{AW} + 8x_{AX} + 11x_{AY} + 9x_{AZ} + 14x_{BW} + 10x_{BX} + 9x_{BY} + 13x_{BZ} + 11x_{CW} + 9x_{CX} + 12x_{CY} + 10x_{CZ} + 13x_{DW} + 11x_{DX} + 10x_{DY} + 12x_{DZ}$

subject to $x_{AW} + x_{AX} + x_{AY} + x_{AZ} = 1$
$x_{BW} + x_{BX} + x_{BY} + x_{BZ} = 1$

$x_{CW} + x_{CX} + x_{CY} + x_{CZ} = 1$
$x_{DW} + x_{DX} + x_{DY} + x_{DZ} = 1$
$x_{AW} + x_{BW} + x_{CW} + x_{DW} = 1$
$x_{AX} + x_{BX} + x_{CX} + x_{DX} = 1$
$x_{AY} + x_{BY} + x_{CY} + x_{DY} = 1$
$x_{AZ} + x_{BZ} + x_{CZ} + x_{DZ} = 1$

## Mixed exercise 2

**1** Bring-it – Depot (326)
Collect-it – Airport (318)
Fetch-it – Docks (317)
Haul-it – Station (321)
Cost £1282

**2 a** One of the following allocations:

| | | | |
|---|---|---|---|
| J – Br (20) | J – C (14) | J – Br (20) | J – Bu (19) |
| K – Bu (19) *or* | K – Bu (19) *or* | K – Cr (14) *or* | K – Cr (14) |
| L – Ba (17) | L – Ba (17) | L – Ba (17) | L – Ba (17) |
| M – C (15) | M – Br (21) | M – Bu (20) | M – Br (21) |

**b** 71 seconds
**c** An allocation not given in part **a** above.

**3 a** Alf – Kitchen (12)
Betty – Gallery (14)
Charlie – Bedroom (14)
Donna – Dining Room (15)
Eve – Grand Hall (14)
Minimum time 69 minutes

**b** Two solutions

| | | |
|---|---|---|
| Alf – Dining Room (19) | | Alf – Dining Room (19) |
| Betty – Hall (12) | | Betty – Bedroom (18) |
| Charlie – Gallery (18) | *or* | Charlie – Gallery (18) |
| Donna – Kitchen (21) | | Donna – Kitchen (21) |
| Eve – Bedroom (20) | | Eve – Hall (14) |

Maximum time 90 minutes

**4 a** There are 4 chauffers but only 3 tasks
**b** *D* – Party (459)   *E* – Dummy
*F* – Film (350)   *G* – Award (231)
Cost £1040

**5** Blue – Maintenance (634)
Green – Post (674)
Orange – Cleaning (825)
Red – Catering (635)
Teal – Copying (554)
Yellow – Computer (530)
Total cost = £3902

**6** Ghost train – Coffee shop (365)
Log flume – Cafe (874)
Roller coaster – Snack shop (665)
Teddie's adventure – Restaurant (794)
Profit £2698

**7** *A* – 3   *B* – Dummy
*C* – 5   *D* – 2
*E* – 1   *F* – 4
Profit £757

**8** Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$
where $i \in \{P, Q, R, S\}$
$j \in \{A, B, C, D\}$
Maximise $C = 13x_{PA} + 17x_{PB} + 15x_{PC} + 18x_{PD} + 15x_{QA} + 19x_{QB} + 12x_{QC} + 19x_{QD} + 16x_{RA} + 20x_{RB} + 13x_{RC} + 22x_{RD} + 14x_{SA} + 15x_{SB} + 17x_{SC} + 14x_{SD}$
subject to $\Sigma x_{Pj} = 1 \quad \Sigma x_{iA} = 1$
$\Sigma x_{Qj} = 1 \quad \Sigma x_{iB} = 1$
$\Sigma x_{Rj} = 1 \quad \Sigma x_{iC} = 1$
$\Sigma x_{Sj} = 1 \quad \Sigma x_{iD} = 1$

**9** Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$
where $i \in \{P, Q, R, S\}$
$j \in \{1, 2, 3, 4\}$
The objective function is
Minimise $P = 143x_{P1} + 243x_{P2} + 247x_{P3} + 475x_{P4} + 132x_{Q1} + 238x_{Q2} + 1000x_{Q3} + 437x_{Q4} + 126x_{R1} + 207x_{R2} + 197x_{R3} + 408x_{R4} + 1000x_{S1} + 222x_{S2} + 238x_{S3} + 445x_{S4}$
subject to $\Sigma x_{Pj} = 1 \quad \Sigma x_{i1} = 1$
$\Sigma x_{Qj} = 1 \quad \Sigma x_{i2} = 1$
$\Sigma x_{Rj} = 1 \quad \Sigma x_{i3} = 1$
$\Sigma x_{Sj} = 1 \quad \Sigma x_{i4} = 1$

**Challenge**

Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$
where $i \in \{W, X, Y, Z\}$
$j \in \{A, B, C, D, E, F, G, H\}$
The objective function is
Minimise $P = 35x_{WA} + 41x_{WB} + 28x_{WC} + 52x_{WD} + 100x_{WE} + 51x_{WF} + 74x_{WG} + 48x_{WH} + 51x_{XA} + 39x_{XB} + 40x_{XC} + 55x_{XD} + 42x_{XE} + 50x_{XF} + 63x_{XG} + 54x_{XH} + 38x_{YA} + 45x_{YB} + 39x_{YC} + 50x_{YD} + 48x_{YE} + 47x_{YF} + 65x_{YG} + 50x_{YH} + 47x_{ZA} + 100x_{ZB} + 48x_{ZC} + 51x_{ZD} + 45x_{ZE} + 53x_{ZF} + 64x_{ZG} + 52x_{ZH}$
subject to $\Sigma x_{Wj} \leqslant 3 \quad \Sigma x_{iA} = 1 \quad \Sigma x_{iE} = 1$
$\Sigma x_{Xj} \leqslant 3 \quad \Sigma x_{iB} = 1 \quad \Sigma x_{iF} = 1$
$\Sigma x_{Yj} \leqslant 3 \quad \Sigma x_{iC} = 1 \quad \Sigma x_{iG} = 1$
$\Sigma x_{Zj} \leqslant 3 \quad \Sigma x_{iD} = 1 \quad \Sigma x_{iH} = 1$

## CHAPTER 3

### Prior knowledge check

**a** *B*, *E*, *F*
**b** A traffic jam could still occur at *D* if, for example, all of the traffic leaving *D* heads along *DB* rather than *DE*.

### Exercise 3A

**1 a** Flow into *B* = flow out of *B*   $w = 3$
Flow into *A* = flow out of *A*   $x = 4$
Flow into *E* = flow out of *E*   $y = 4$
Flow into *D* = flow out of *D*   $z = 13$
**b** Feasible flow = 28
**c** *CE* and *ED* are saturated.
**d** *BD* has capacity 8.
**e** Along *SAT* the current flow is 8.

**2 a** Flow into *A* = flow out of *A*   $w = 9$
Flow into *E* = flow out of *E*   $x = 5$
Flow into *C* = flow out of *C*   $y = 2$
Flow into *D* = flow out of *D*   $14 = y + x + z$
$\Rightarrow 14 = 2 + 5 + z$
$\Rightarrow z = 7$
**b** Feasible flow = 38
**c** *BE* and *AC* are saturated.
**d** Flow along *SD* is 14.
**e** Flow along *SBET* is 15.

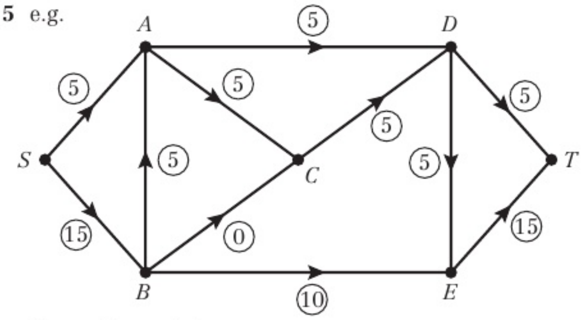
**3 a** Source vertex is *F*
**b** Sink vertex is *C*
**c** Flow into *A* = flow out of *A*   $w = 8$
Flow into *B* = flow out of *B*   $x = 3$
Flow into *D* = flow out of *D*   $y = 20$
Flow into *G* = flow out of *G*   $z = 4$
**d** Feasible flow = 27

- **e** Saturated arcs are *AC*, *FC*, *FG*
- **f** Capacity of *FB* is 8

**4 a** Source vertex is *E*

- **b** Sink vertex is *C*
- **c** Flow into *A* = flow out of *A* $w = 5$
  Flow into *B* = flow out of *B* $x = 3$
  Flow into *G* = flow out of *G* $y = 4$
  Flow into *D* = flow out of *D* $z = 5$
- **d** Feasible flow = 20
- **e** Saturated arcs are *BA*, *ED*, *DG*, *GF*
- **f** Flow along *FC* = 11

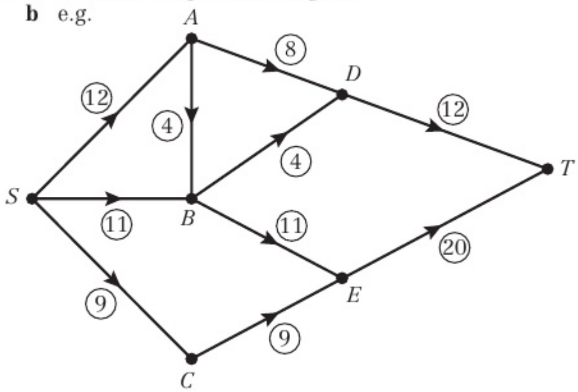
**5** e.g.




Many other solutions.

**6 a** All flows are positive integers.

- **b** e.g.




Many other solutions.

- **c** 11

## Exercise 3B

**1** Cut $C_1 = 19 + 8 + 10 + 4 = 41$
Cut $C_2 = 20 + 7 + 20 = 47$

**2** Cut $C_1 = 7 + 9 + 4 + 19 = 39$
Cut $C_2 = 15 + 3 + 16 = 34$

**3** Cut $C_1 = 15 + 45 + 18 + 15 + 10 = 103$
Cut $C_2 = 15 + 10 + 20 + 10 + 15 + 8 = 78$
Cut $C_3 = 20 + 45 + 18 + 15 + 8 = 106$

**4** Cut $C_1 = 16 + 16 + 4 + 25 = 61$
Cut $C_2 = 30 + 6 + 23 + 10 = 69$

**5** Cut $C_1 = 30 + 32 + 18 + 30 = 110$
Cut $C_2 = 20 + 50 + 18 + 35 = 123$
Cut $C_3 = 20 + 15 + 8 + 15 + 10 + 18 + 14 = 100$

**6 a** A cut, in a network with source *S* and sink *T*, is a set of arcs whose removal separates the network into two parts, *X* and *Y*, where *X* contains at least *S* and *Y* contains at least *T*.

- **b i** 42
  - **ii** Cut $C_1 = 14 + 14 + 4 + 18 = 50$
    Cut $C_2 = 24 + 14 + 4 + 18 = 60$

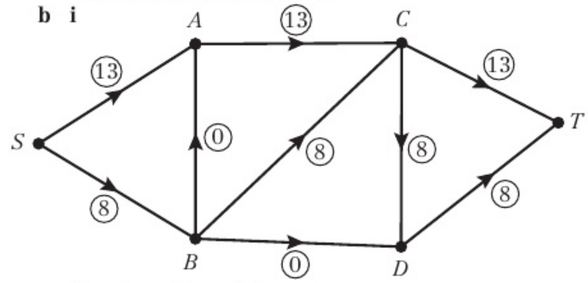
**7 a** $x = 8$

- **b** The flow in the arc with capacity $y$ is out of the cut and so $y$ is not related to the capacity of the cut.

## Exercise 3C

**1 a** Max flow along *SACT* = 13
Max flow along *SBCDT* = 8

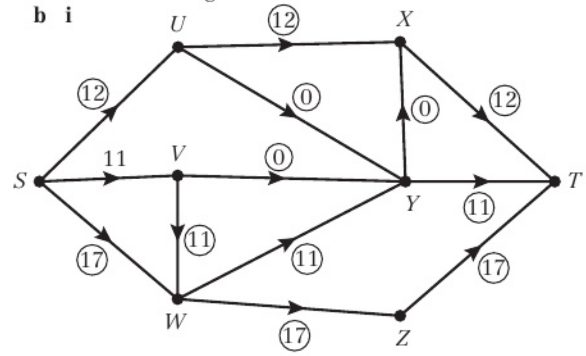
- **b i**




  - **ii** Value of initial flow = 21

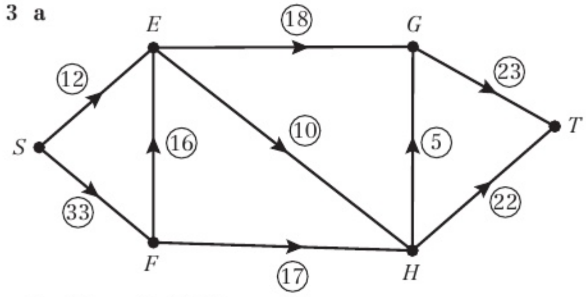
**2 a** Max flow along *SUXT* = 12
Max flow along *SWZT* = 17
Max flow along *SVWYT* = 11

- **b i**




  - **ii** Value of initial flow = 40

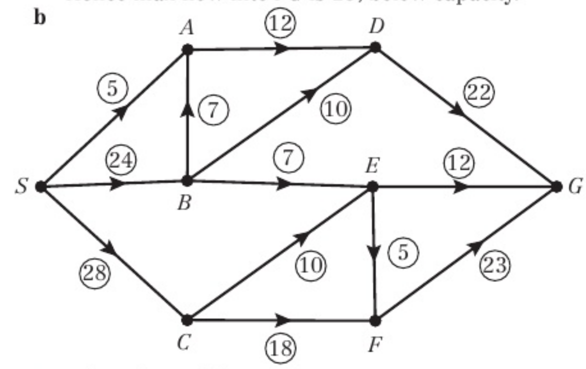
**3 a**




- **b** Value of initial flow = 45

**4 a** The arcs into *F* are *CF* and *EF*, capacity total 23. Hence max flow into *FG* is 23, below capacity.

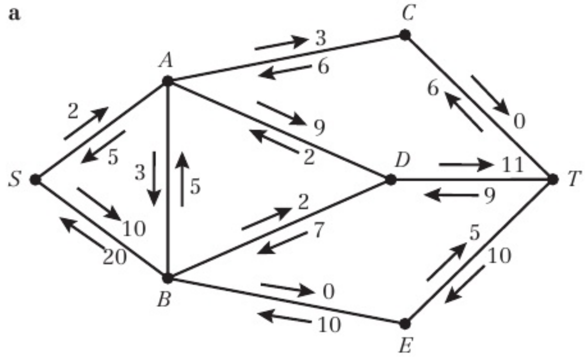
- **b**




- **c** Value of initial flow = 57

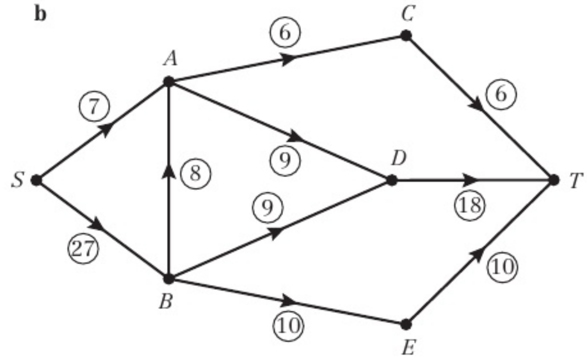
## Exercise 3D

**1 a**



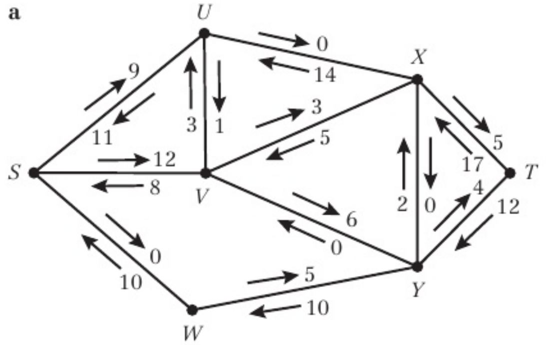

e.g. – there are many other combinations of flows possible
*SBADT* – 5  *SADT* – 2  *SBDT* – 2

**b**



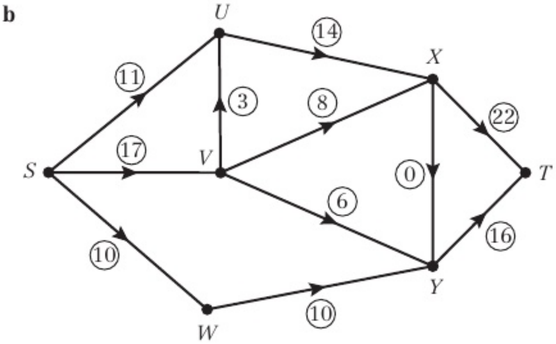

Value of maximum flow = 34

**2 a**



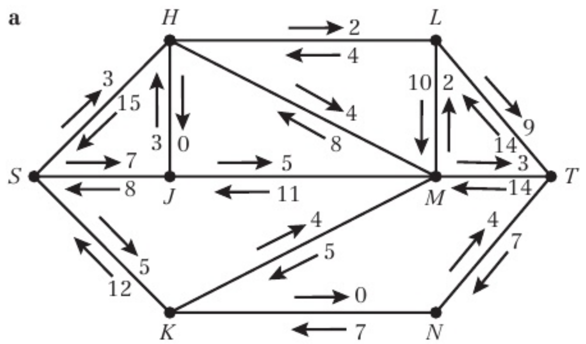

e.g. – there are many other combinations of flows possible
*SVYT* – 4  *SVXT* – 3  *SVYXT* – 2

**b**



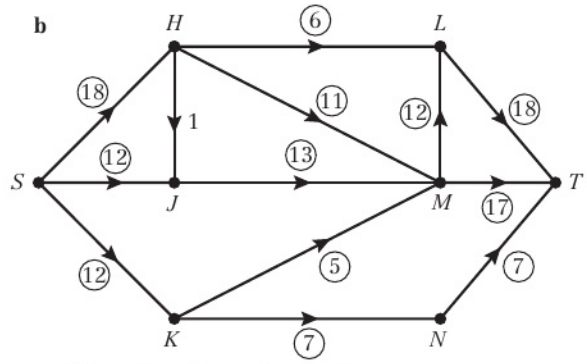

Value of maximum flow is 38

**3 a**



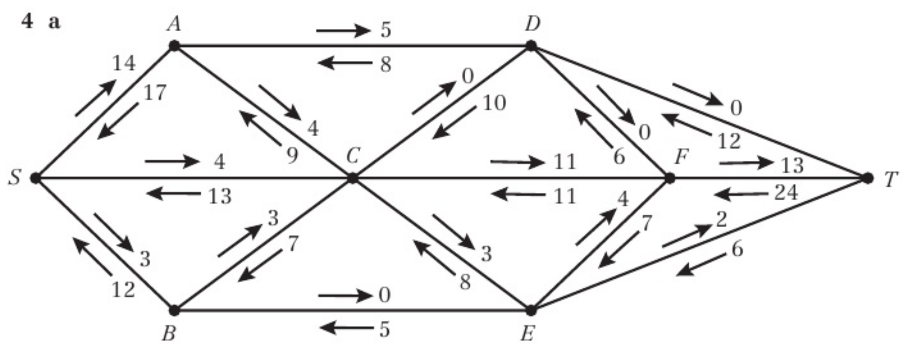

e.g. – there are many other combinations of flows possible
*SHMT* – 3  *SJMLT* – 2  *SJHLT* – 2

**b**



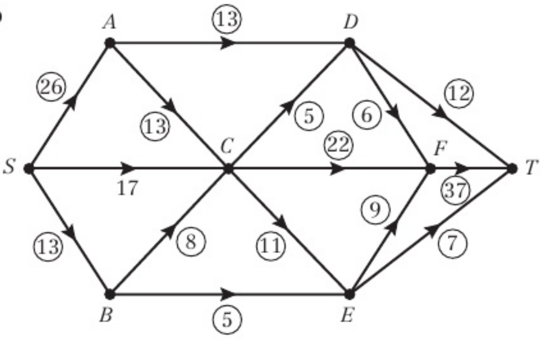

Value of maximum flow is 42

**4 a**




e.g. – there are many other combinations of flows possible
*SACFT* – 4
*SADCFT* – 5
*SCFT* – 2
*SCEFT* – 2
*SBCET* – 1

**b**




Value of maximum flow is 56

**5 a** $C_1$: 7 + 7 + 6 = 20   $C_2$: 8 + 8 + 7 = 23

**b** *SBDET* (2) or *SBCET* (2) or *SBDEFT* (2) or *SBCEFT* (2)
Maximum flow = 20

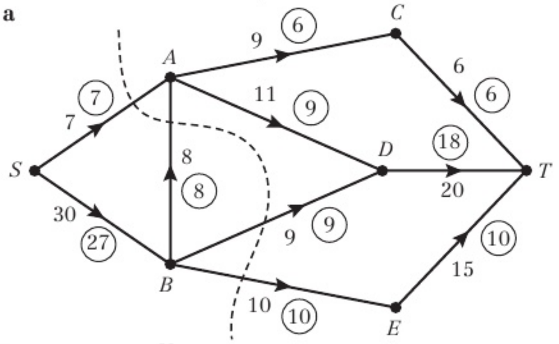
**6 a** *SACFEGT*,   *SBEGT*

**b** *SBEGT*
*SACFEGT* is not valid since the package cannot travel from *F* to *E*.

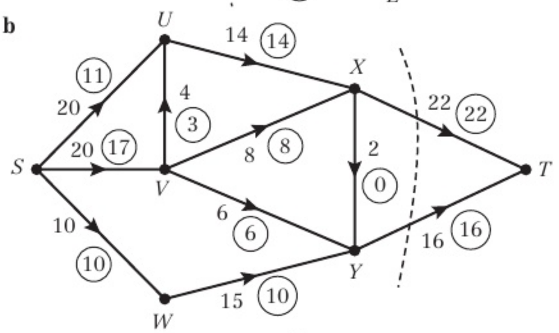
## Exercise 3E

**1** To prove that the flow is maximal using the maximum flow–minimum cut theorem, it is sufficient to find a cut with capacity equal to the flow. Each diagram shows a suitable cut, but there may be others.

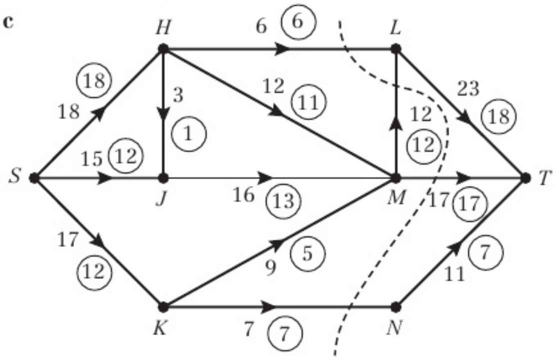
**a**



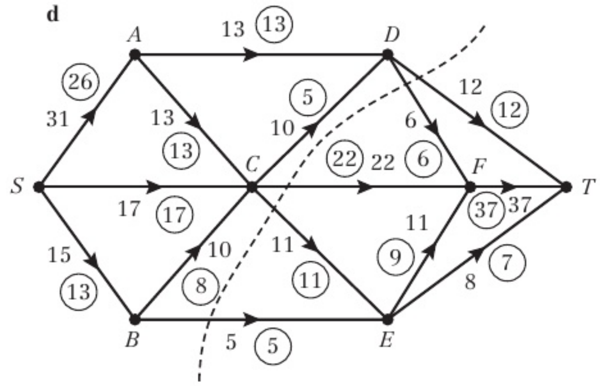

**b**




**c**




**d**




**2 a** $x = 12, y = 8$

**b** 37

**c** Maximum flow = 41
Flow-augmenting route
[1] *SAEHT* (2) and *SBCGT* (2)
or
[2] *SABCGT* (2) and *SBAEHT* (2)

**d** Using [1] Minimum cut through:
*HT*, *EF*, *BF*, *BG*, *BC*, *SC*
Value of min cut = 10 + 2 + 4 + 2 + 8 + 15 = 41
or
Using [2] Minimum cut through:
*HT*, *FT*, *FG*, *BG*, *BC*, *SC*
Value of min cut = 10 + 3 + 3 + 2 + 8 + 15 = 41
By minimum cut – maximum flow theorem, the flow found in *C* is maximal.

**3 a** The saturated arcs are: *SA*, *AD*, *DE*, *ET*, *EG*

**b** 57 cars per minute

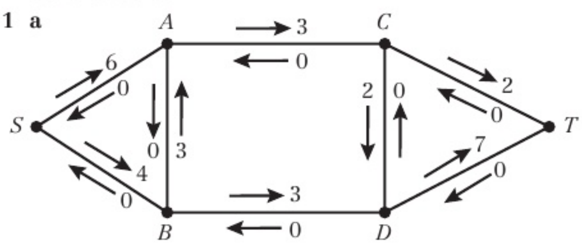
**c** $C_1$: 95 cars per minute
$C_2$: 67 cars per minute

**d** *SDCFT*

**e** Augmented flow = 63
Cut through *FT*, *ET* and *EG* has capacity
38 + 16 + 9 = 63
By the maximum flow–minimum cut theorem, the flow is maximal.

## Mixed exercise 3

**1 a**

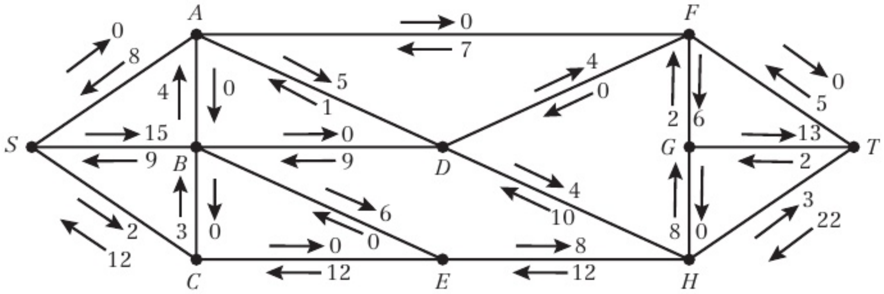



e.g. – many different flow combinations
*SBDT* – 3
*SACT* – 2
*SBACDT* – 1
Value of flow = 6

**b** Cut through *AC* and *BD* minimum cut = 6 so by maximum flow–minimum cut theorem, flow is maximum.

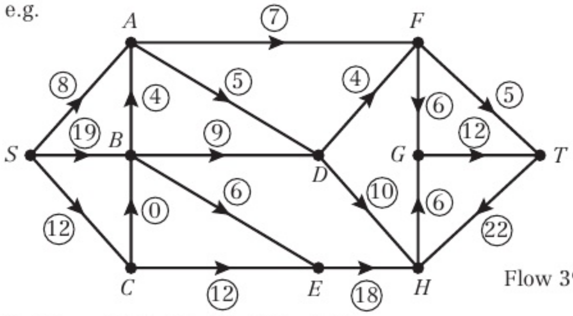
**2 a** e.g. traffic moving through a one-way system of roads.

**b**




e.g. *SBEHGT* – 6 and *SBADFGT* – 4
or *SBADHGT* – 4 and *SCBEHT* – 2 and *SBEHGT* – 4 etc.

**c** e.g.



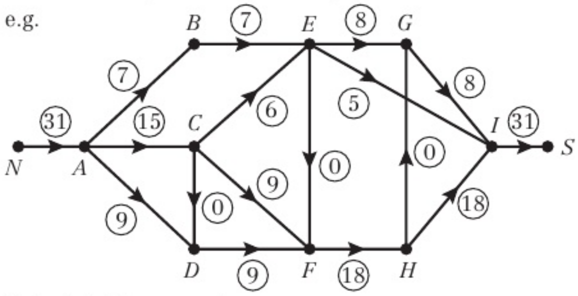

Flow 39

**d** Cut through *SA*, *BA*, *BD*, *BE* and *CE*

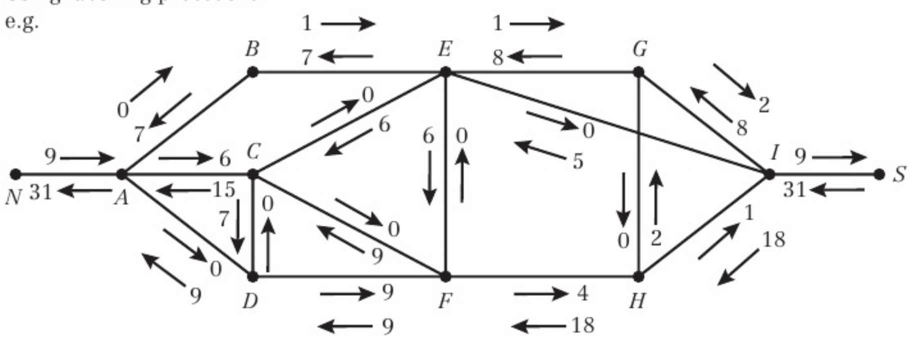
**e** The arcs are saturated.

**3 a** *A*, *F*, *G* and *H*, possible flow in > possible flow out

**b** e.g.

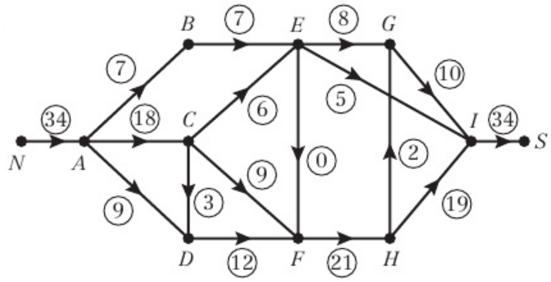



**c** Using labelling procedure
e.g.




| | | | |
|---|---|---|---|
| e.g. | | | |
| If *HI* = 16 | If *HI* = 17 | If *HI* = 18 | If *HI* = 19 |
| *NACDFHIS* – 3 | *NACDFHGIS* – 1 | *NACDFHGIS* – 2 | *NACDFHGIS* – 1 |
| | *NACDFHIS* – 2 | *NACDFHIS* – 1 | *NACDFEGIS* – 2 |
| | | | Final flow 34 |

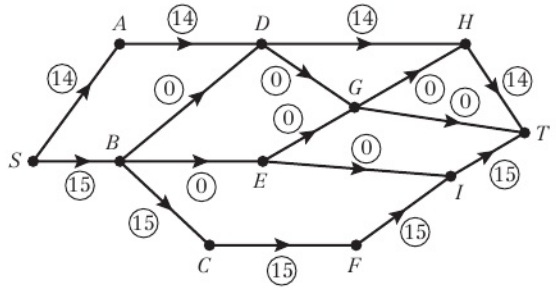
**d**



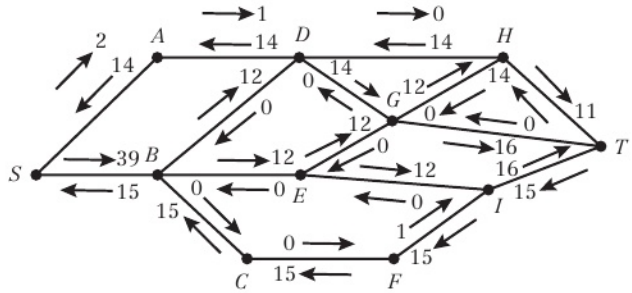

**e** and **f** cuts go through either *GI*, *EI* and *HI or AB*, *CE*, *EF*, *HG* and *HI*

**4 a i** Flow along *SBCFIT* = 15 **ii** Flow along *SADHT* = 14

**b**

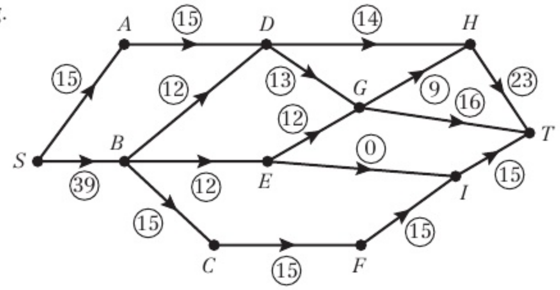



**c**




e.g. *SADGT* – 1
*SBDGT* – 12
with *SBEIT* – 12 or *SBEGHT* – 9
and *SBEGT* – 3
giving a total flow of 54

**d** e.g.

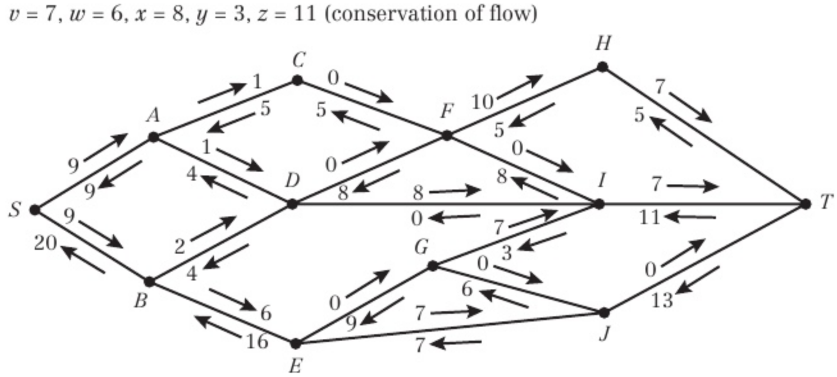



**e** Cuts passing through *AD*, *BD*, *BE*, *BC* and *AD*, *BD*, *BE*, *CF* both total 54, so by the maximum flow–minimum cut theorem, the flow is maximal.

**f** The flow into *D* and into *C* cannot increase so increase the flow along *BE*.

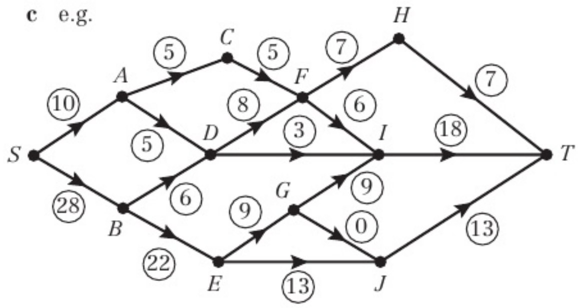
**5 a** $v = 7$, $w = 6$, $x = 8$, $y = 3$, $z = 11$ (conservation of flow)

**b**




Increasing flow by an additional 3
e.g. *SBDIT* – 2
*SADIT* – 1
Additional flow increases (reversing initial flow)
e.g. *SBEJGIT* – 4
*SBEJGIFHT* – 2

**c** e.g.




**d** The capacity of a cut is the sum of the capacities of those arcs crossed by the cut and that flow into the cut.

**e** **i** 12 + 18 + 13 = 43

**ii** Cut through *CF*, *AD*, *BD*, *BE* has capacity 38.

**iii** The value of the minimum cut is 38, which is equal to the flow given. Hence, by the maximum flow–minimum cut theorem, the flow is maximal.

**6** **a** A cut, in a network with source *S* and sink *T*, is a set of arcs whose removal separates the network into two parts, *X* and *Y*, where *X* contains at least *S* and *Y* contains at least *T*.

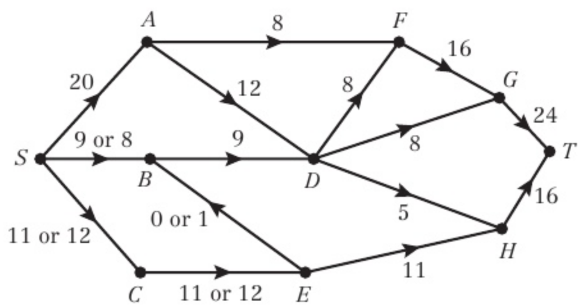
**b** $C_1 = 40$  $C_2 = 56$

**c** Max flow = Min cut = 40

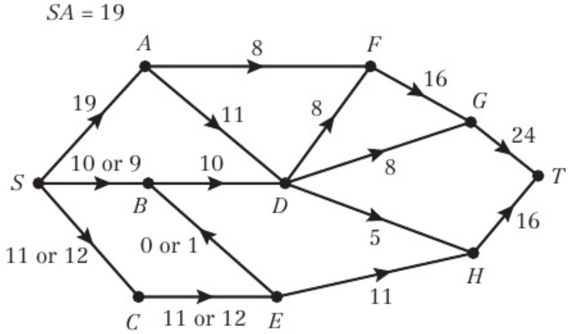
**d** e.g. Flow out of *F* = flow into *F* = 16
Flow along *DG* = 8
Flow out of *G* = flow into *G* = 16 + 8 = 24
So flow along *GT* is 24.

**e** e.g. Flow into *A* = flow out of *A*
So flow along $AD \leqslant 12$
Flow into *D* = flow out of *D* = 21
So flow along *AD* + flow along *BD* = 21
So flow along *AD* and *BD* could be 12 + 9 or 11 + 10
So possible flows are 20 and 19

**f** *SA* = 20




*SA* = 19




**g** There are 2 more – *CE* could be 11 or 12 in each case.

**Challenge**

**a** 50 amps

**b** There is no cut that can be made that passes through only saturated, or empty, arcs. This is due to the added restriction that the maximum current flow through any component is 25 amps.

## CHAPTER 4

## Prior knowledge check

**a** 36

**b** A saturated arc has a flow equal to its capacity.

**c** For the flow to be maximal, there must be a cut that crosses saturated arcs. None of the arcs are saturated.

**d** The value of the flow is 36. Since it isn't maximal, there can be no cut of this value. By the maximum flow–minimum cut theorem, the minimum cut must have a value greater than 36.

## Exercise 4A

**1** **a** *AB*: 7, *BC*: 4, *BD*: 3

**b** *PQ*: 7, *SQ*: 8, *QR*: 15

**c** *AB*: 10, *BD*: 4, *BE*: 6, *CD*: 4, *DF*: 8, *EF*: 6, *FG*: 14

**2** **a** $C_1$: 20 + 16 − 6 + 12 = 42, $C_2$: 10 + 12 + 17 = 39

**b** The maximum flow is at most 39.

**c**

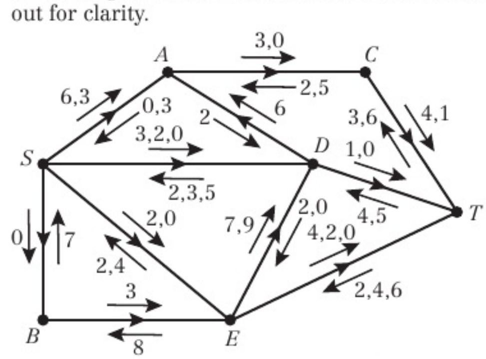
| Arc | Flow |
|---|---|
| *SB* | 12 |
| *AB* | 9 |
| *BD* | 11 |
| *BE* | 8 |

**3** **a** Consider vertex *C*. The minimum flow into *C* is 21 + 14 = 35.
The maximum flow out of *C* is 27 < 35, so no feasible flow.

**b** **i** *CE*  **ii** 35

**4** **a** *DT* = 14, *AC* = 11, *CT* = 11, *SB* = 15, *BE* = 15

**b** In the diagram, old values have not been crossed out for clarity.



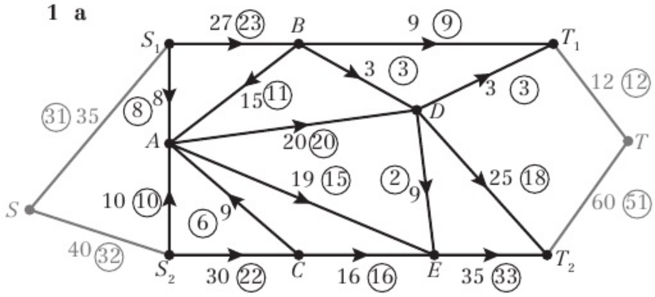

| Augmenting path | Increase |
|---|---|
| *SACT* | 3 |
| *SDT* | 1 |
| *SDET* | 2 |
| *SET* | 2 |

Total increase = 8
Maximum flow = initial flow + augmented flow = 37 + 8 = 45

**c** Consider a cut through *AC*, *DT* and *ET* (all saturated)
Value of the cut = 14 + 15 + 16 = 45
Since a cut has been found with value equal to the flow, the flow must be maximal.

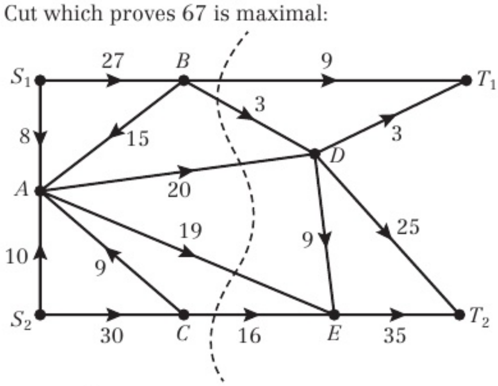
## Exercise 4B

**1 a**

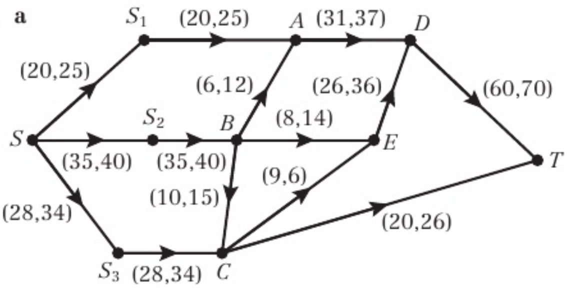



**b** Multiple maximal flow patterns. Total flow = 67

**c** Cut which proves 67 is maximal:

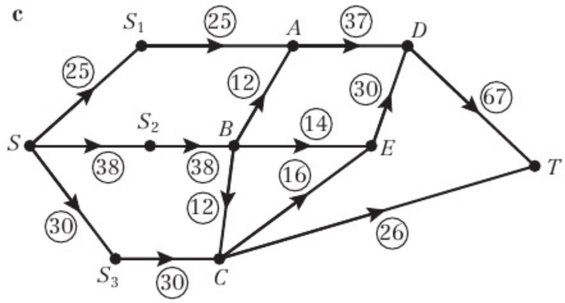



**2 a**



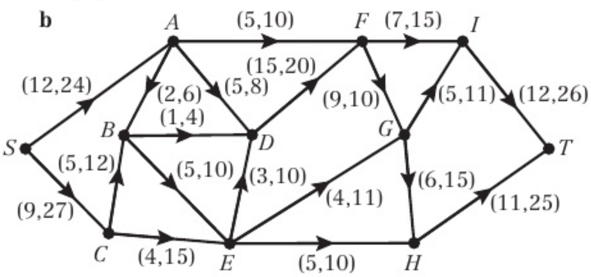

**b** 93

**c**

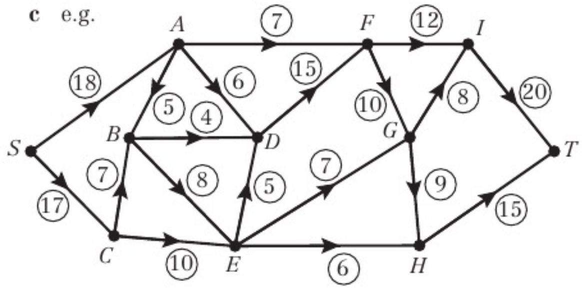



**3 a** Source vertices are $A$ and $C$. Sink vertices are $H$ and $I$.

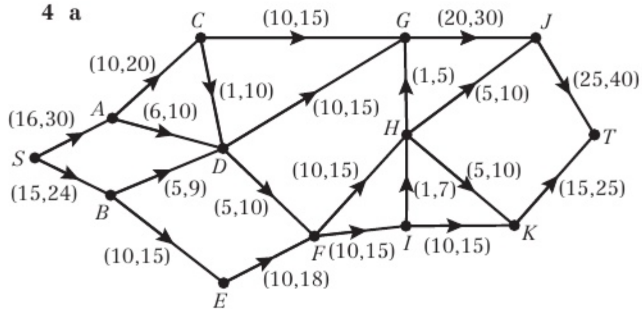
**b**




**c** e.g.




**4 a**




**b** 60

**c** The maximum flow out is at most 60.

**d**

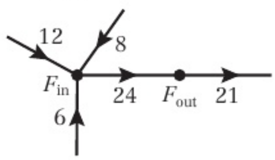
| Augmenting path | Increase |
|---|---|
| *SACGJT* | 2 |
| *SBEFIKT* | 1 |
| *SBDFIKT* | 1 |
| *SBDGHJT* | 1 |

Maximum flow = 49 + 5 = 54

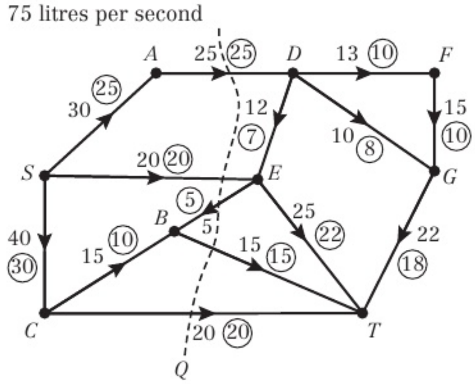
**e** The cut through *AC*, *AD*, *BD*, and *BE* has value 20 + 10 + 9 + 15 = 54
Value of cut = value of flow, so flow is maximal.

## Exercise 4C

**1**

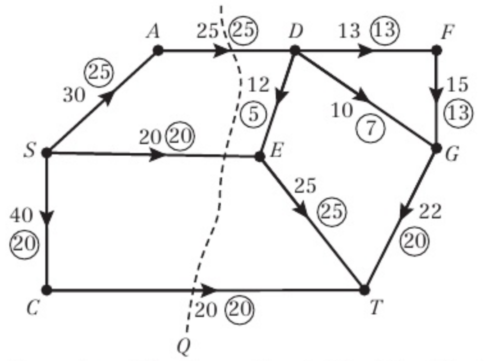



**2 a** 75 litres per second



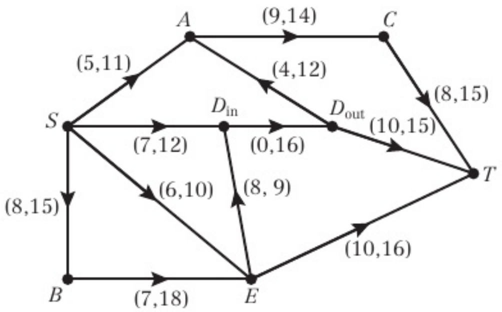

The value of the cut $Q$ is 25 + 20 + 15 + 20 = 75 litres per second. Since this matches the flow, the flow must be maximal.

**b**

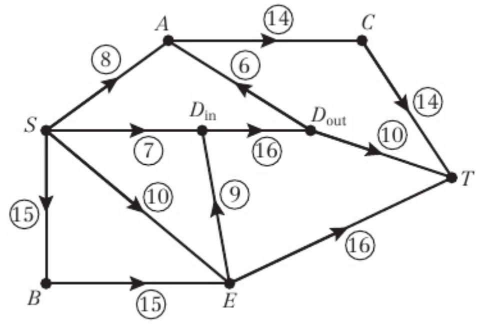



**c** New value of the flow pattern is 25 + 20 + 20 = 65 litres per second. The new value of the cut is also 25 + 20 + 20 = 65 litres per second. This proves that this flow is maximal.

**3 a**

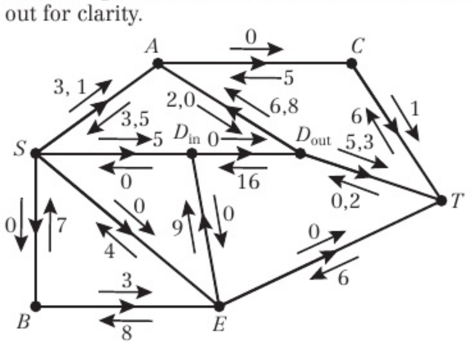



**b**




Flow = 40

**c** In the diagram, old values have not been crossed out for clarity.




| Augmenting path | Increase |
|---|---|
| $SAD_{out}T$ | 2 |

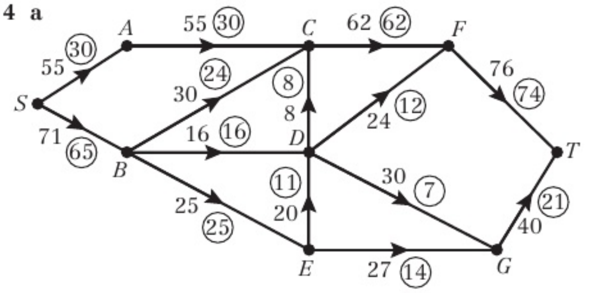
New maximum flow = 40 + 2 = 42

**d** One possible cut goes through $AC$, $AD_{out}$, $D_{in}D_{out}$, and $ET$.

Value of cut = 14 − 4 + 16 + 16 = 42

Value of cut = maximum flow found in part **c**

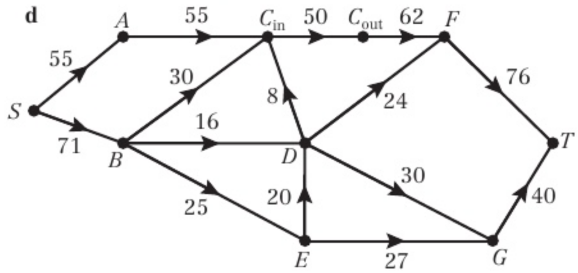
**4 a**




**b** 103

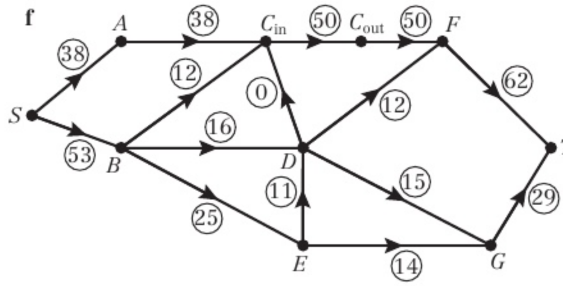
**c** The arcs $CF$, $BD$ and $BE$ are all saturated.
A cut through these arcs has capacity
62 + 16 + 25 = 103
By the minimum cut – maximum flow theorem, the maximum flow is 103.

**d**



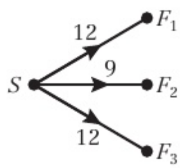

**e** 91

**f**



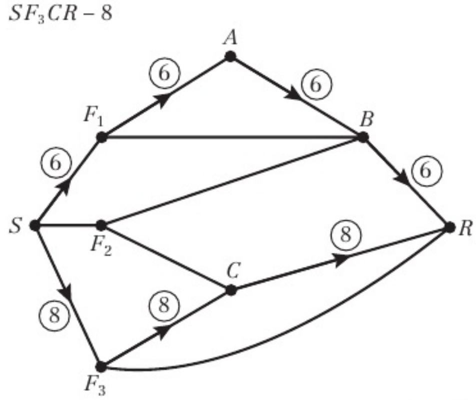

## Mixed exercise 4

**1 a**

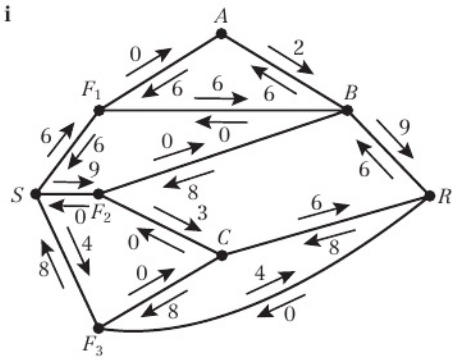



**b i** $SF_1ABR - 6$

**ii** $SF_3CR - 8$



**c** **i**




e.g.

| | |
|---|---|
| $SF_1BR$ – 6 | $SF_3R$ – 4 |
| $SF_2BR$ – 3 | $SF_2BR$ – 6 |
| $SF_2CR$ – 3 | $SF_2CR$ – 3 |
| $SF_3R$ – 4 | $SF_1BR$ – 3 |

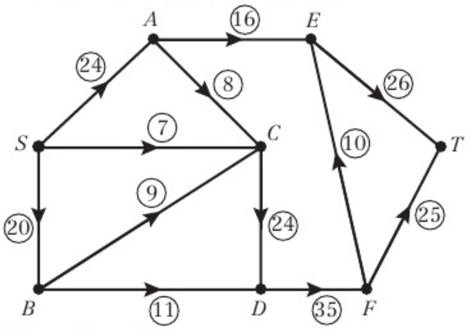
Total flow 30

**ii** For the first augmenting set given, a cut through $BR$, $F_2C$, $F_3C$ and $F_3R$ goes through saturated arcs and has value 30. By the maximum flow–minimum cut theorem, the flow is maximal.

**2** **a** $C_1$: 70, $C_2$: 65

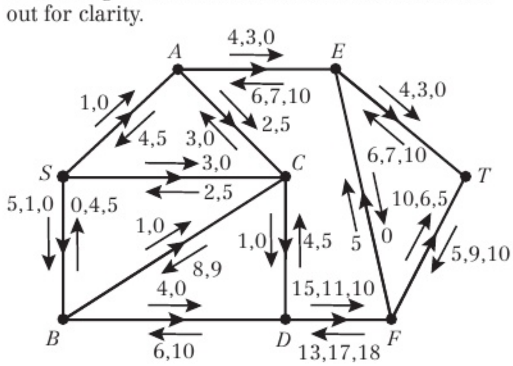
**b** The maximum flow is at most 65.

**c**




Flow value = 51

**d** In the diagram, old values have not been crossed out for clarity.




| Augmenting path | Increase |
|---|---|
| *SAET* | 1 |
| *SCAET* | 3 |
| *SBDFT* | 4 |
| *SBCDFT* | 1 |
| **Total** | 9 |

Maximum flow = 51 + 9 = 60

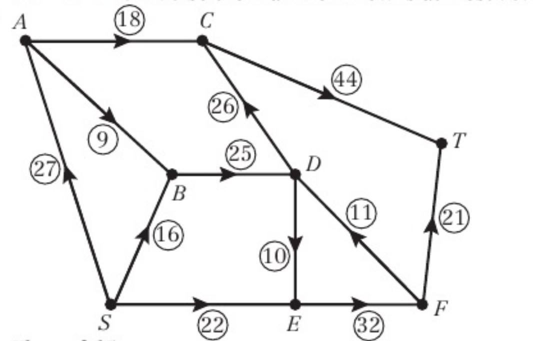
**e** A cut with value 60 passes through $SA$, $SC$, $BC$ and $BD$.

**3** **a** The maximum flow out of $C$ is 44.
The minimum flow into $C$ is 44 so the only feasible flow through $C$ occurs when the flows through $AC$ and $CD$ are at their minimum values.

**b** 9

**c** 44 – 11 + 40 = 73 so the maximum flow is at most 73.

**d**

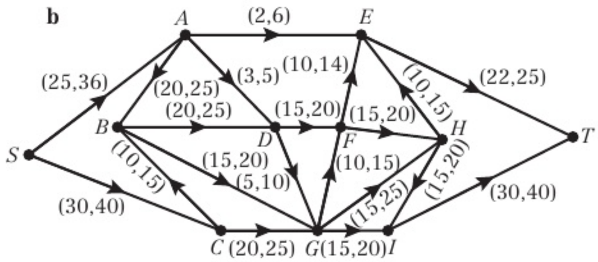



Flow of 65

**e** Augmenting path is $SEFT$ with value 4.
Maximum flow = 65 + 4 = 69
$CT$ and $FT$ are the saturated arcs, giving a minimum cut of 44 + 25 = 69.

**4** **a** Sources: $A$ and $C$, sinks: $E$ and $I$

**b**

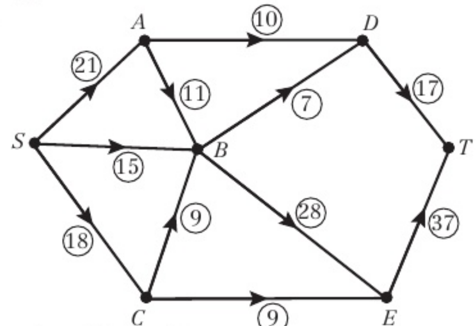



**c** 6 + 5 + 25 + 20 – 10 = 46 so the maximum flow is at most 46.

**d** Minimum flow into $D$ = 20 + 3 = 23
Maximum flow out of $D$ = 10, so not feasible.

**e** 13

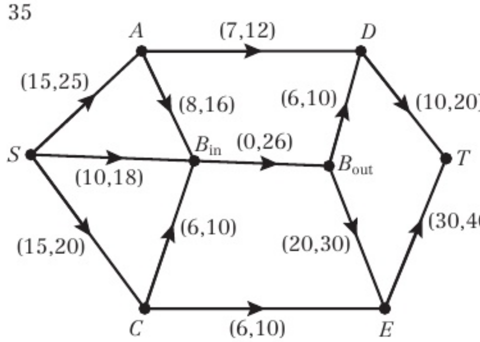
**5** **a**




Value of flow = 54

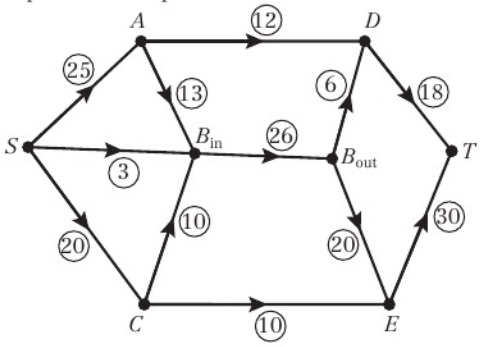
**b** 35

**c**



**d** 48

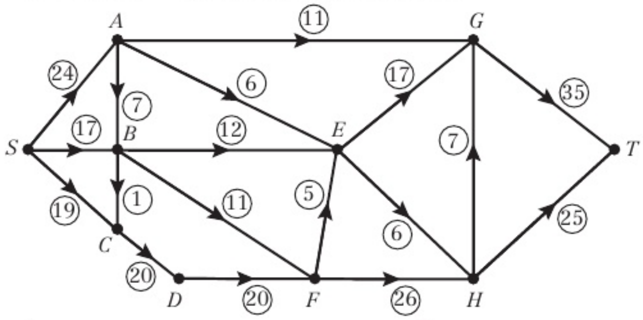
A possible flow pattern is



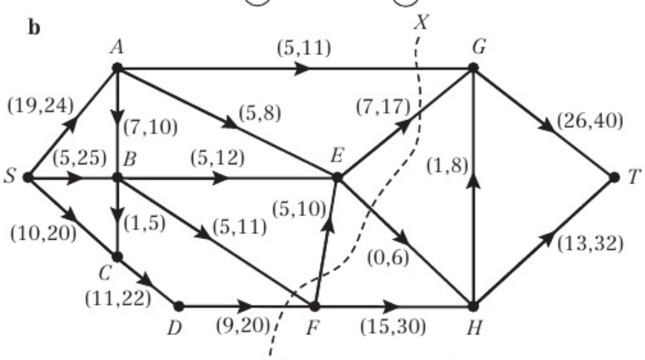

**e** A cut through $AD$, $B_{in}B_{out}$, and $CE$ has capacity 12 + 26 + 10 = 48. This is equal to the flow found in part **d**, so is maximal.

**Challenge**

**a** Max flow = 60. A possible flow pattern is

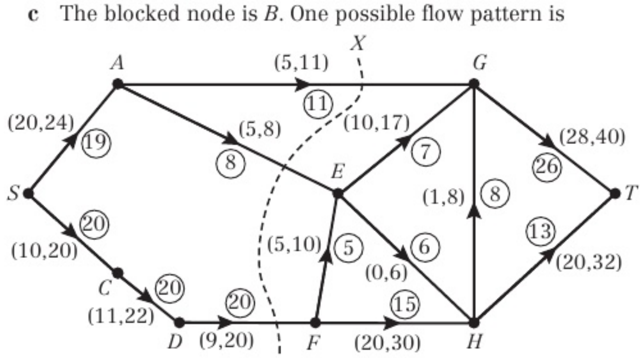



**b**




The value of the cut, $X$, is 11 + 17 + 6 − 5 + 11 + 20 = 60. Since value of cut = value of flow, the flow must be maximal.

**c** The blocked node is $B$. One possible flow pattern is




**d** The value of the cut, $X$, is 11 + 8 + 20 = 39, which matches the flow, so the flow of 39 is maximal.

## Review exercise 1

**1 a**

| | $W_1$ | $W_2$ | $W_3$ | Supply |
|---|---|---|---|---|
| $F_1$ | 2 | 2 | | 4 |
| $F_2$ | | 3 | | 3 |
| $F_3$ | | 4 | 4 | 8 |
| **Demand** | 2 | 9 | 4 | |

Cost $2 \times 7 + 2 \times 8 + 3 \times 2 + 4 \times 6 + 4 \times 3$
$= 14 + 16 + 6 + 24 + 12 = 72$

**b** Shadow costs and improvement indices:

| | | 7 | 8 | 5 |
|---|---|---|---|---|
| | | $W_1$ | $W_2$ | $W_3$ |
| 0 | $F_1$ | × | × | 1 |
| −6 | $F_2$ | 8 | × | 5 |
| −2 | $F_3$ | 0 | × | × |

**c** There are no negative improvement indices and so given solution is optimal and gives minimum cost. If there was a negative improvement index then using that route may reduce cost.

**2 a** The total supply is greater than the total demand.

**b** The solution would otherwise be degenerate.

**c** Solution is:

| | $J$ | $K$ | $L$ |
|---|---|---|---|
| $A$ | | | 9 |
| $B$ | 8 | | 5 |
| $C$ | 1 | 11 | |

Cost = 167

Shadow costs and improvement indices:

| | | 8 | 13 | 0 |
|---|---|---|---|---|
| | | $J$ | $K$ | $L$ |
| 0 | $A$ | 4 | 2 | × |
| 0 | $B$ | × | 4 | × |
| −4 | $C$ | × | × | 4 |

No negative improvement indices so solution is optimal.

**3 a**

| | $D$ | $E$ | $F$ |
|---|---|---|---|
| $A$ | 20 | 4 | |
| $B$ | | 26 | 6 |
| $C$ | | | 14 |

**b** $S(A) = 0$ $S(B) = -1$ $S(C) = 7$
$D(D) = 21$ $D(E) = 24$ $D(F) = 18$
$I_{AF} = 16 - 0 - 18 = -2$
$I_{BD} = 18 + 1 - 21 = -2$
$I_{CD} = 15 - 7 - 21 = -13$*
$I_{CE} = 19 - 7 - 24 = -12$

**c**

| | $D$ | $E$ | $F$ | |
|---|---|---|---|---|
| $A$ | $20 - \theta$ | $4 + \theta$ | | entering cell $CD$ |
| $B$ | | $26 - \theta$ | $6 + \theta$ | $\theta = 14$ |
| $C$ | $\theta$ | | $14 - \theta$ | exiting cell $CF$ |

| | $D$ | $E$ | $F$ | |
|---|---|---|---|---|
| $A$ | 6 | 18 | | |
| $B$ | | 12 | 20 | Cost £1384 |
| $C$ | 14 | | | |

**4 a** e.g.

| | *D* | *E* | *F* |
|---|---|---|---|
| ***A*** | 6 | | |
| ***B*** | 0 | 5 | |
| ***C*** | | 4 | 4 |

or

| | *D* | *E* | *F* |
|---|---|---|---|
| ***A*** | 6 | 0 | |
| ***B*** | | 5 | |
| ***C*** | | 4 | 4 |

Cost £470

**b** Shadow costs and improvement indices:

| | | 20 | 40 | 50 |
|---|---|---|---|---|
| | | *D* | *E* | *F* |
| 0 | ***A*** | × | × | −40 |
| −10 | ***B*** | 10 | × | 0 |
| −20 | ***C*** | 10 | × | × |

or

| | | 20 | 30 | 40 |
|---|---|---|---|---|
| | | *D* | *E* | *F* |
| 0 | ***A*** | × | 10 | −30 |
| 0 | ***B*** | × | × | 0 |
| −10 | ***C*** | 0 | × | × |

**c** Optimal solution either

| | *D* | *E* | *F* |
|---|---|---|---|
| ***A*** | 2 | | 4 |
| ***B*** | | 5 | |
| ***C*** | 4 | 4 | |

or

| | *D* | *E* | *F* |
|---|---|---|---|
| ***A*** | 2 | | 4 |
| ***B*** | 4 | 1 | |
| ***C*** | | 8 | |

Cost of both solutions is £350.

**5 a**

| | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|
| $F_1$ | 20 | 15 | |
| $F_2$ | | 10 | 15 |
| $F_3$ | | | 15 |

**b** $I_{13} = 11 - 0 - 7 = 4$
$I_{21} = 12 - 1 - 10 = 1$
$I_{31} = 9 - 0 - 10 = -1$
$I_{32} = 6 - 0 - 4 = 2$
Since $I_{31}$ is negative, pattern is not optimal.

**c**

| | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|
| $F_1$ | 10 | 25 | |
| $F_2$ | | | 25 |
| $F_3$ | 10 | | 5 |

**d** $I_{13} = 11 - 0 - 8 = 3$ $I_{21} = 12 - 0 - 10 = 2$
$I_{22} = 5 - 0 - 4 = 1$ $I_{32} = 6 + 1 - 4 = 3$
No negative improvement indices so the solution is optimal.
Cost 525 units.

**6 a** Idea of many supply and demand points and many units to be moved. Costs are variable and dependent upon the supply and demand points. The aim is to minimise costs.

**b** Supply = 120 Demand = 110 so not balanced

**c**

| | *D* | *E* | *F* |
|---|---|---|---|
| ***A*** | 45 | | |
| ***B*** | 5 | 30 | |
| ***C*** | | 30 | 10 |

Cost 545

**d** Shadow costs and improvement indices:

| | | 5 | 7 | 3 |
|---|---|---|---|---|
| | | *D* | *E* | *F* |
| 0 | ***A*** | × | −4 | −3 |
| −1 | ***B*** | × | × | −2 |
| −3 | ***C*** | 0 | × | × |

**e**

| | *D* | *E* | *F* |
|---|---|---|---|
| ***A*** | 15 | 30 | |
| ***B*** | 35 | | |
| ***C*** | | 30 | 10 |

Cost 425

**7 a**

| | *P* | *Q* | *R* | *S* | **Supply** |
|---|---|---|---|---|---|
| ***A*** | 6 | | | | 6 |
| ***B*** | 1 | 5 | 1 | | 7 |
| ***C*** | | | 5 | 0 | 5 |
| ***D*** | | | | 8 | 8 |
| **Demand** | 7 | 5 | 6 | 8 | |

Total cost of initial solution is £175

**b** Shadow costs and improvement indices:

| | | 5 | 2 | 3 | 5 |
|---|---|---|---|---|---|
| | | *P* | *Q* | *R* | *S* |
| 0 | ***A*** | × | 4 | 5 | 5 |
| 4 | ***B*** | × | × | × | 2 |
| 4 | ***C*** | −1 | 3 | × | × |
| 3 | ***D*** | 2 | 6 | 2 | × |

The entering cell is *CP*.

| | *P* | *Q* | *R* | *S* |
|---|---|---|---|---|
| ***A*** | 6 | | | |
| ***B*** | $1 - \theta$ | 5 | $1 + \theta$ | |
| ***C*** | $\theta$ | | $5 - \theta$ | 0 |
| ***D*** | | | | 8 |

Setting $\theta = 1$, *BP* becomes the exiting cell.
New solution is

| | *P* | *Q* | *R* | *S* | |
|---|---|---|---|---|---|
| ***A*** | 6 | | | | 6 |
| ***B*** | | 5 | 2 | | 7 |
| ***C*** | 1 | | 4 | 0 | 5 |
| ***D*** | | | | 8 | 8 |
| | 7 | 5 | 6 | 8 | |

Total costs is now £174.

Shadow costs and improvement indices:

| | | 5 | 3 | 4 | 6 |
|---|---|---|---|---|---|
| | | ***P*** | ***Q*** | ***R*** | ***S*** |
| 0 | ***A*** | × | 3 | 4 | 4 |
| 3 | ***B*** | 1 | × | × | 2 |
| 3 | ***C*** | × | 3 | × | × |
| 2 | ***D*** | 3 | 6 | 2 | × |

There are no negative improvement indices, so the solution is optimal.

**8** **a** Let $x_{ij}$ be the number of coaches to be moved from $i$ to $j$, where $i \in \{A, B, C\}$ and $j \in \{D, E, F\}$ and $x_{ij} \geqslant 0$.

**b** Minimise distance $S = 40x_{AD} + 70x_{AE} + 25x_{AF} + 20x_{BD} + 40x_{BE} + 10x_{BF} + 35x_{CD} + 85x_{CE} + 15x_{CF}$

**c** Constraints:
$x_{AD} + x_{AE} + x_{AF} = 8$ (no. of coaches at depot $A$)
$x_{BD} + x_{BE} + x_{BF} = 5$ (no. of coaches at depot $B$)
$x_{CD} + x_{CE} + x_{CF} = 7$ (no. of coaches at depot $C$)
$x_{AD} + x_{BD} + x_{CD} = 4$ (no. of coaches at depot $D$)
$x_{AE} + x_{BE} + x_{CE} = 10$ (no. of coaches at depot $E$)
$x_{AF} + x_{BF} + x_{CF} = 6$ (no. of coaches at depot $F$)
Number of coaches at $A$, $B$ and $C$ = number of coaches at $D$, $E$ and $F$.

**9** Let $x_{ij}$ be number of units transported from $i$ to $j$ where $i \in \{W, X, Y\}$ (warehouse) and $j \in \{J, K, L\}$ (supermarket) and $x_{ij} \geqslant 0$

Minimise $C = 3x_{WJ} + 6x_{WK} + 3x_{WL} + 5x_{XJ} + 8x_{XK} + 4x_{XL} + 2x_{YJ} + 5x_{YK} + 7x_{YL}$

subject to
$x_{WJ} + x_{WK} + x_{WL} = 34$
$x_{XJ} + x_{XK} + x_{XL} = 57$
$x_{YJ} + x_{YK} + x_{YL} = 25$
$x_{WJ} + x_{XJ} + x_{YJ} = 20$
$x_{WK} + x_{XK} + x_{YK} = 56$
$x_{WL} + x_{XL} + x_{YL} = 40$
$x_{ij} \geqslant 0 \quad i \in \{W, X, Y\}$ and $j \in \{J, K, L\}$

**10** Let $x_{ij}$ be the number of units transported from $i$ to $j$, in 1000 litres where $i \in \{F, G, H\}$ and $j \in \{S, T, U\}$ and $x_{ij} \geqslant 0$
Supply is not equal to demand, so the problem is unbalanced.

Minimise $C = 23x_{FS} + 31x_{FT} + 46x_{FU} + 35x_{GS} + 38x_{GT} + 51x_{GU} + 41x_{HS} + 50x_{HT} + 63x_{HU}$

subject to
$x_{FS} + x_{FT} + x_{FU} \leqslant 540$
$x_{GS} + x_{GT} + x_{GU} \leqslant 789$
$x_{HS} + x_{HT} + x_{HU} \leqslant 673$
$x_{FS} + x_{GS} + x_{HS} \geqslant 257$
$x_{FT} + x_{GT} + x_{HT} \geqslant 348$
$x_{FU} + x_{GU} + x_{HU} \geqslant 410$

**11** $A$–$I$, $B$–$H$, $C$–$P$, $D$–$S$ *or* $A$–$S$, $B$–$I$, $C$–$P$, $D$–$H$
(both £1068)

**12** Machine 1 – Job 2 (5)
Machine 2 – Job 4 (5)
Machine 3 – Job 3 (3)
Machine 4 – Job 1 (2)
Minimum time: 15 hours

**13** **a** I – $C$, II – $A$, III – $B$, IV – $D$
**b** 69 minutes

**14** **a** $A$–$Q$, $B$–$T$, $C$–$S$, $D$–$U$, $E$–$R$, $F$–$P$ **b** £101

**15** **a** A dummy column is needed so that the number of tasks matches the number of workers. This is a requirement of the Hungarian algorithm.
**b** Beth – no task
Josh – task 2
Louise – task 3
Oliver – task 1
**c** 51 minutes

**16** **a** Bill – $B$
Steve – $D$
Jo – $C$
No mechanic is allocated to centre $A$
**b** £1255

**17** $D$–$A$, $K$–$M$, $H$–$S$, $T$–$L$ *or* $D$–$A$, $K$–$L$, $H$–$S$, $T$–$M$ *or* $D$–$M$, $K$–$A$, $H$–$S$, $T$–$L$ *or* $D$–$S$, $K$–$A$, $H$–$M$, $T$–$L$
Total 88 points

**18** **a** $C$–III, $J$–IV, $N$–II, $S$–I *or* $C$–III, $J$–I, $N$–II, $S$–IV
83 minutes so earliest ending time is 11:23 am
**b** Subtracting all entries from some $n \geqslant 36$
e.g. subtracting from 36

| | I | II | III | IV |
|---|---|---|---|---|
| $C$ | 24 | 2 | 8 | 20 |
| $J$ | 23 | 4 | 0 | 24 |
| $N$ | 21 | 4 | 4 | 22 |
| $S$ | 25 | 3 | 0 | 26 |

**19** **a** $A$–2 $B$–4 $C$–3 $D$–1 *or*
$A$–3 $B$–4 $C$–1 $D$–2
**b** £1 160 000
**c** Gives other solution not given in part **a**.

**20** **a** Setting the time for Charles to complete stage 3 as 50 minutes.
Reducing rows:

| | | | |
|---|---|---|---|
| 2 | 0 | 4 | 1 |
| 2 | 0 | 3 | 1 |
| 2 | 0 | 30 | 1 |
| 1 | 0 | 3 | 1 |

Reducing columns:

| | | | |
|---|---|---|---|
| 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 27 | 0 |
| 0 | 0 | 0 | 0 |

Amy – stage 2, Bob – stage 3, Charles – stage 4, Davina – stage 1 *or* Amy – stage 4, Bob – stage 3, Charles –stage 2, Davina – stage 1
**b** Fastest time 81 minutes

**21** Minimise $C = 20x + 26y + 36z$
subject to $2(y + z) \leqslant x$
$x + z \leqslant 9y$
$x + z \geqslant 4y$
$2z \geqslant y$
$x \geqslant 0, y \geqslant 0, z \geqslant 0$
$x + y + z \geqslant 250$

**22 a** $C_1 = 7 + 14 + 0 + 14 = 35$ $C_2 = 7 + 14 + 5 = 26$ $C_3 = 8 + 9 + 6 + 8 = 31$

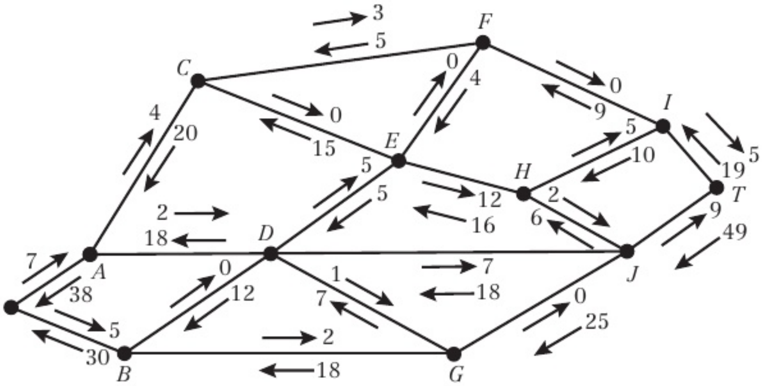
**b** The minimum cut has a value of 26, which is equal to the flow. Hence, by the maximum flow–minimum cut theorem, the flow is maximal.

**c** Using *EJ* (capacity 5) will increase flow by 1 – i.e. increase it to 27 since only one more unit can leave *E*.
Using *FH* (capacity 3) will increase flow by 2 – i.e. increase it to 28 since only two more units can leave F.
Thus choose option 2 add *FH* capacity 3.

**23 a** Using conservation of flow through vertices $x = 16$ and $y = 7$

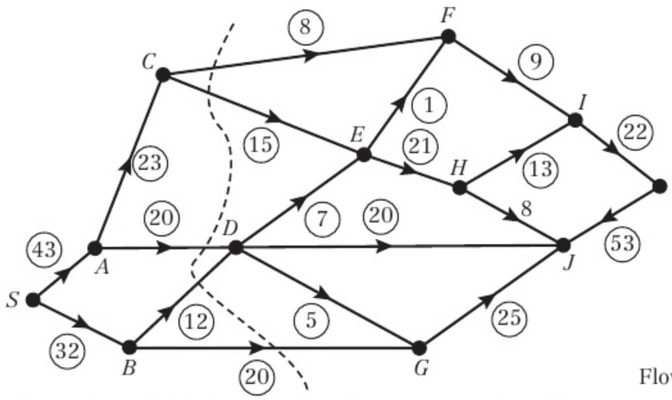
**b** $C_1 = 86$, $C_2 = 81$

**c**




Augmenting paths: *SADEHJT* – 2 *SACFEHIT* – 3 *SBGDJT* – 2

**d**



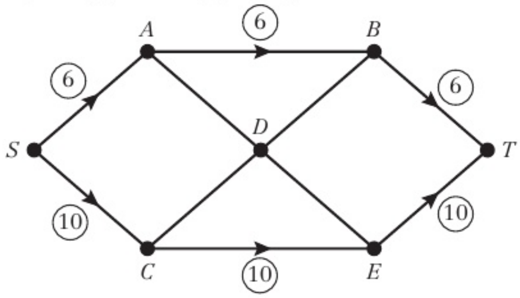

Flow 75

**e** The cut through *CF*, *CE*, *AD*, *BD*, *BG* has capacity 75. By the maximum flow–minimum cut theorem, the flow is maximal.

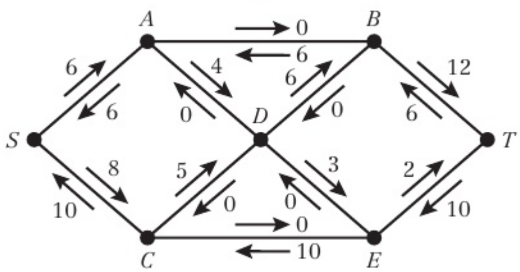
**24 a** Finds a cut less than 30 giving its value e.g. cut through *AB*, *AD*, *CD*, *CE* (25) or *AB*, *BD*, *ET* (24) *or* a consideration of flow input/flow output through *A* and *C*

**b** **i** *SABT* (6) **ii** *SCET* (10)

**c**

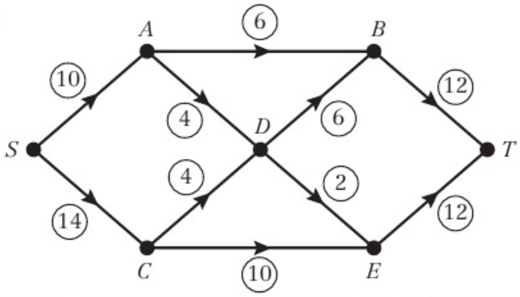



**d**




| Augmenting path | Increase |
|---|---|
| *SADBT* | 4 |
| *SCDET* | 2 |
| *SCDBT* | 3 |
| **Total** | 8 |

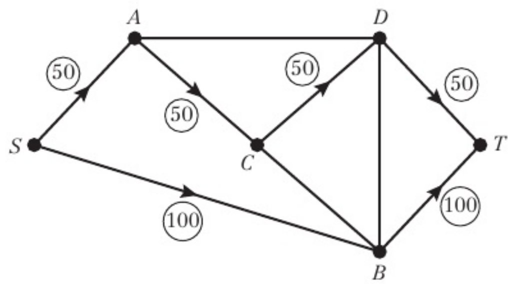
**e** e.g.



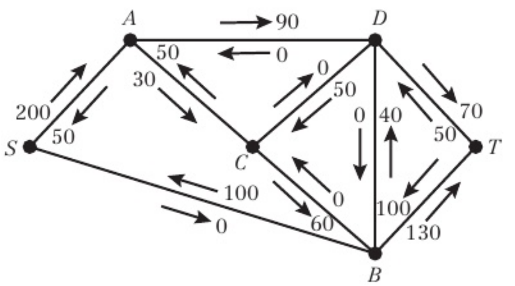

**f** The cut though *AB*, *BD*, *ET* has capacity 24. By the maximum flow–minimum cut theorem, the flow is maximal.

**25 a i** Maximum flow along *SACDT* = 50 **ii** Maximum flow along *SBT* = 100

**b**

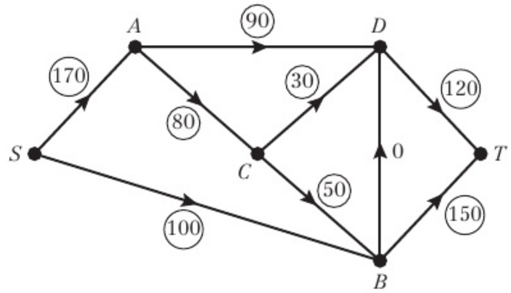



**c**



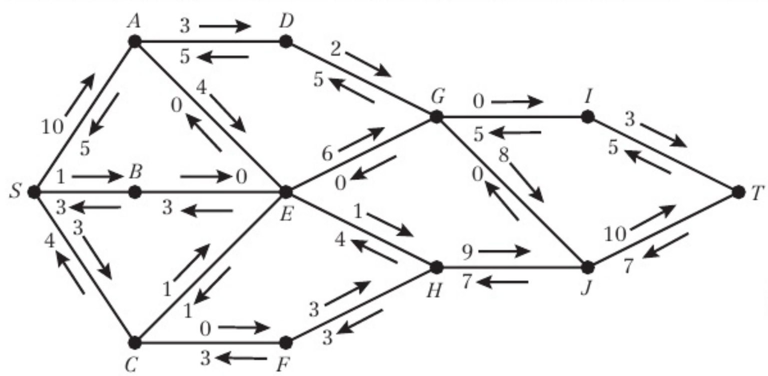

Augmenting paths:
*SADT* – 70
*SACBT* – 30
*SADCBT* – 20
maximum flow 270

**d**



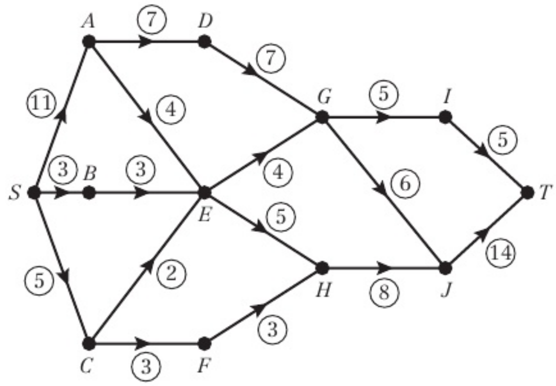

**e** The cut through *AD*, *AC*, *SB* has capacity 270. By the maximum flow–minimum cut theorem, the flow is maximal.

**26 a**




Augmenting paths:
e.g. *SADGJT* flow 2
*SAEGJT* flow 4
*SCEHJT* flow 1

**b**




**c** 19 students

**d** Cut through *DG*, *AE*, *BE*, *CE*, *CF* has capacity 19. By maximum flow–minimum cut theorem, flow is maximised.

**27 a** $x = 12, y = 7$

**b** 41

**c**


**d** For example *SBEDT* (1)
*SCBFT* (2)
*SCBEDT* (1)

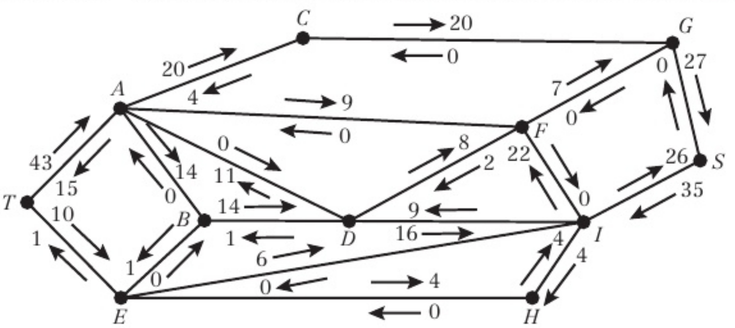
Maximum flow = 41 + 4 = 45

**e** The cut through *SA, BA, BE, BF* and *CF* has capacity 45. By the maximum flow – minimum cut theorem, the flow is maximal.

**28 a** $x = 9, y = 16$

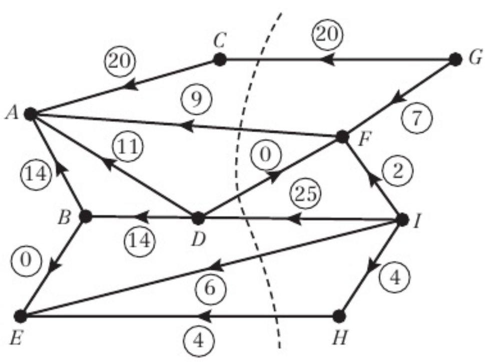
**b** Initial flow = 53. There are not enough saturated arcs for a minimum cut so the flow is not maximal.

**c**




Augmenting paths:
e.g. *SIDAT* – 9
*SIFDAT* – 2
Maximum flow – 64

**d**




**e** The cut through *CG*, *AF*, *DF*, *DI*, *IE*, *EH* has capacity 64. By the maximum flow–minimum cut theorem, the flow is maximal.

**29 a** *SD* 10, *CT* 19, *BD* 6. Flow = 36

**b**

| Augmenting path | Increase |
|---|---|
| *SBET* | 1 |
| *SDBET* | 2 |

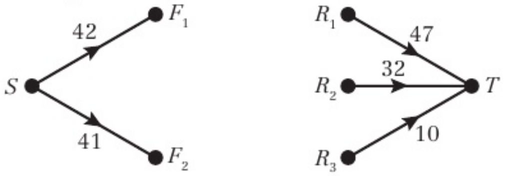
Max flow = 36 + 3 = 39

**c** Cut through *SA*, *SD* and *SB* has capacity 12 + 12 + 15 = 39

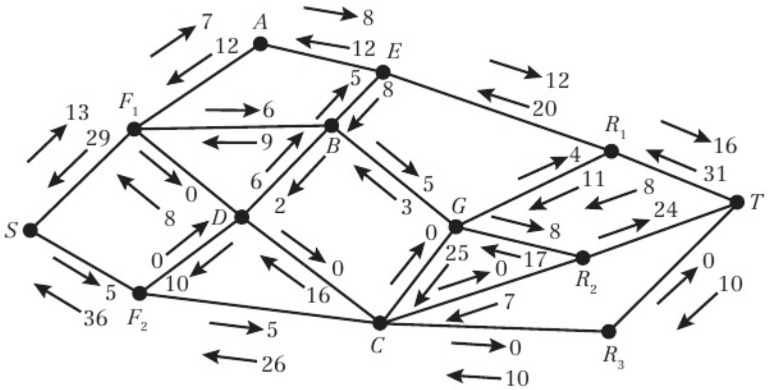
By maximum flow–minimum cut theorem, flow is maximised.

**30 a** $x = 3$, $y = 26$

**b**




**c**




Augmenting paths:

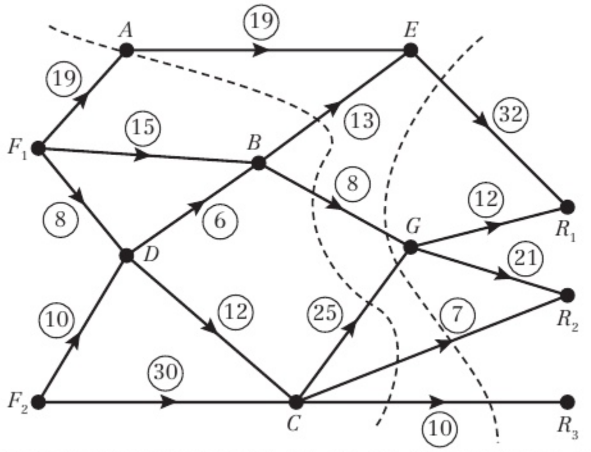
e.g. $SF_1AER_1T - 7$

$SF_1BER_1T - 5$

$SF_1BGR_1T - 1$

$SF_2CDBGR_2T - 4$

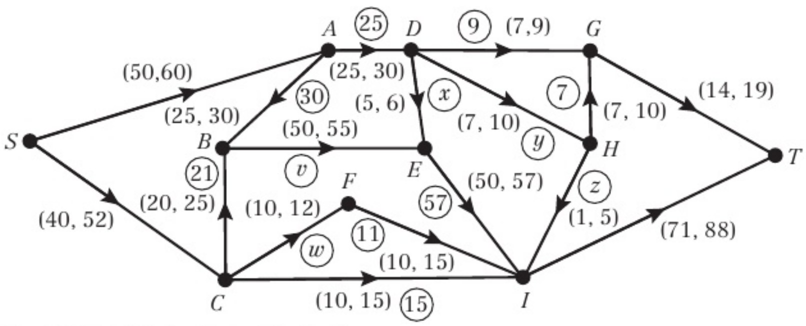
**d** e.g.




Max flow 82

**e** Minimum cut either through $ER_1$, *BG*, *CG*, $CR_2$, $CR_3$ or through *AE*, *BE*, *BG*, *CG*, $CR_2$, $CR_3$ has capacity 82. By the maximum flow–minimum cut theorem, the flow is maximal.

**31 a**



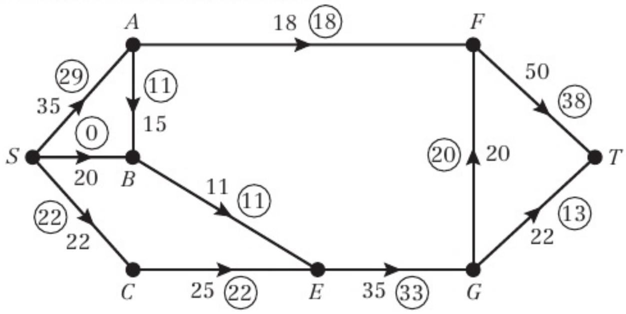

**b** $V = 51$, $W = 11$, $x = 6$, $y = 10$, $z = 3$

**c** 102

**d** Cut through *DG*, *DH*, *EI*, *FI* and *CI* has capacity 102. Flow is maximised by max flow – min cut theorem.

**32** **a** $w = 30$, $x = 21$, $y = 45$, $z = 22$ so flow = 67
Consider cut *AF*, *DF*, *FG*, *GT* = 18 + 7 + 20 + 22 = 67.
So by the maximum flow – minimum cut theorem, the flow is maximal.

**b** One possible flow is shown below

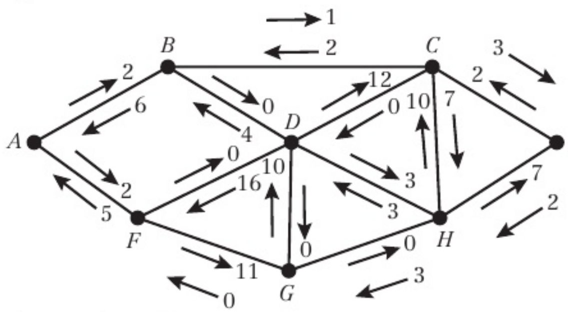



**c** Flow = 51
Cut through *AF*, *BE* and *SC* has capacity 18 + 11 + 22 = 51. Flow is maximised by maximum flow – minimum cut theorem.

**33** **a** *A*, *E* and *G*

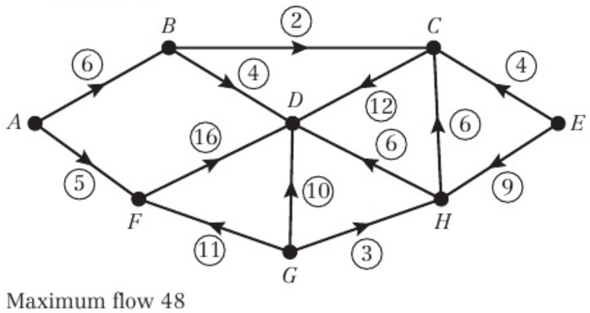
**b** 45

**c**




Augmenting paths:
e.g. *EHD* – 2
*ECHD* – 1

**d**

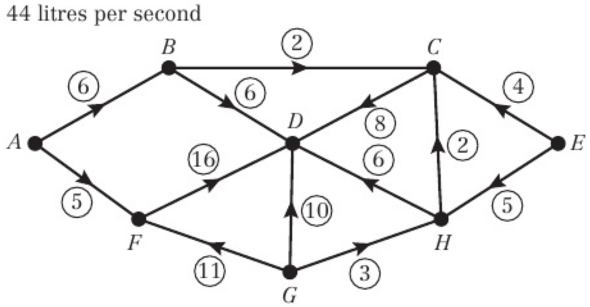



Maximum flow 48

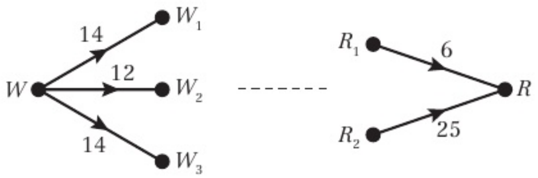
**e** The cut through *DB*, *DC*, *DH*, *DG*, *DF* has value 48. By the maximum flow – minimum cut theorem, the flow is maximal.

**f** 44 litres per second

**g**

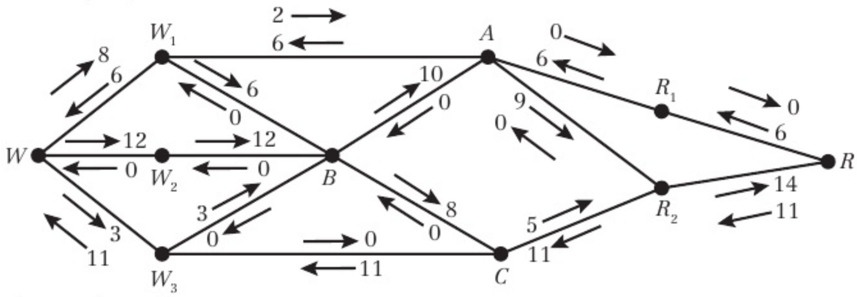


**34 a**

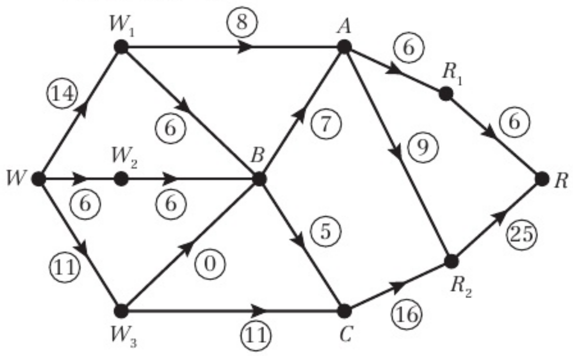



**b** **i** $WW_1AR_1R - 6$
**ii** $WW_3CR_2R - 11$

**c**




Augmenting paths:
e.g. $WW_1BAR_2R - 6$
$WW_1AR_2R - 2$
$WW_2BCR_2R - 5$
$WW_2BAR_2R - 1$




Max flow 31

**d** 12 – For this network, but may be different for others.

**e** Make no use of the opportunity.
All arcs out of $A$ and $C$ are saturated, so the total flow cannot be increased when the number of van loads from $A$ or $C$ to $R_1$ or $R_2$ is increased.

**Challenge**

**1** **a** Total demand (650) is greater than total supply (640).

**b** Optimal solution is 140 boxes $A$ to $R$, 120 boxes $B$ to $P$, 160 boxes $B$ to $Q$, 20 boxes $B$ to $S$, 200 boxes $C$ to $S$ and 10 (non-existent) boxes $D$ to $R$. Cost £2030

**2**

| Worker | Assign to |
|---|---|
| *A* | 1 and 3 |
| *B* | 2 |
| *C* | 4 |
| *D* | 5 |

Minimum cost = £247

**3** **a** Flow = 160
Cut through $AD$, $BD$, $BE$, $CF$ has capacity 160. Flow is maximised by max flow–min cut theorem.

**b** Replace $CF$ to give an extra flow of 9.

**c** Replace $SB$ and $BE$ to give an extra flow of 60.

## CHAPTER 5
### Prior knowledge check

1 a Dijkstra's algorithm
 b *SCT* c *SADET* d *SADE*

### Exercise 5A

1 a Shortest route *SBDHT* length 87
 b Longest route *SAEGT* length 100
2 a Shortest route length is 88 with route *SACGT*
 b Longest route length is 97 with route *SBFHT*
3 a Shortest route length is 93 with route *SBEHT*
 b Longest route length is 101 with route *SCFJT*
4 a *SBFGT*
 b Cost = £84 000
5 The maximum profit is £145 000
The maximum route is *SDFT*
In practical terms the company's strategy is:
Year 1 – advertise on both TV and Radio
Year 2 – advertise on TV only.

### Exercise 5B

1 a Minimax route *SCDHT* or *SCEHT* – both of value 30
 b Maximin route *SCEGT* of value 20
2 a Minimax route *SBEHT* of value 33
 b Maximin route *SAEIT* of value 14
3 a Minimax route *SBEIT* or *SBFIT* – both of value 27
 b Maximin routes *SAEIT*, *SBEIT*, *SBFIT*, *SCEIT*, *SCFIT* all of value 18
4 a Maximin
 b *SADGT* or *SBDGT*
 c 8 feet
5 a Minimax
 b *SBDGT* total length = 44 miles
 or
 *SBDIT* total length = 45 miles

**Challenge**

Multiple possible answers

### Exercise 5C

1 a Minimax
 b Stage – number of years remaining
 State – resorts listed
 Action – resort added to brochure
 c

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| 1 | *BC* | *A* | *ABC* | 60* |
| | *AC* | *B* | *ABC* | 85* |
| | *AB* | *C* | *ABC* | 75* |
| 2 | *A* | *B* | *AB* | Max(65, 75) = 75* |
| | | *C* | *AC* | Max(75, 85) = 85 |
| | *B* | *A* | *AB* | Max(70, 75) = 75 |
| | | *C* | *BC* | Max(65, 60) = 65* |
| | *C* | *A* | *AC* | Max(75, 85) = 85 |
| | | *B* | *BC* | Max(80, 60) = 80* |
| 3 | None | *A* | *A* | Max(55, 75) = 75 |
| | | *B* | *B* | Max(70, 65) = 70* |
| | | *C* | *C* | Max(60, 80) = 80 |

 d *BCA* cost = £7000

2 a Stage – phase being considered
 State – number of days remaining
 Action – number of days allocated
 Destination – number of days remaining
 Value – total costs
 b The minimum cost is £66 000. The time should be allocated as follows:

| Activity | Clearance | Repairing | Modernisation | Decorating |
|---|---|---|---|---|
| Number of days | 5 | 15 | 5 | 5 |

3 The minimum cost is £210 000. The aircraft should be built as follows:

| Month | March | April | May | June |
|---|---|---|---|---|
| Number of aircraft built in each month | 3 | 0 | 3 | 2 |

 b Any part of an optimal path is itself optimal.
 c The minimum cost is £220 000. The aircraft should be built as follows:

| Month | March | April | May | June |
|---|---|---|---|---|
| Number of aircraft built in each month | 1 | 2 | 3 | 2 |

4 The optimum route is: Home – *A* – *E* – *F* – *I* – Home with a value of £2700

### Mixed exercise 5

1 a Minimax route is *SCEGT*, value 22
 b *CEGT*
2 Maximin route is *SBDHT*, value 21
3 **a** and **b** There are two possible courses of action each of value £65

| Product | Butter | Cheese | Yoghurt |
|---|---|---|---|
| Units to be used | 2 | 2 | 1 |

| Product | Butter | Cheese | Yoghurt |
|---|---|---|---|
| Units to be used | 2 | 3 | 0 |

4 Tracing back there are two routes
*SC*, *CF*, *FT* ⇒ *SCFT*
*SA*, *AD*, *DT* ⇒ *SADT*
Maximum altitude on these routes is 40 (×100 ft) = 4000 ft

5 a

| Stage | State | Action | Cost | Total cost |
|---|---|---|---|---|
| 2 | 0 | *A*<br>*C* | 2<br>3 | 2<br>3* |
| | 1 | *A*<br>*B*<br>*C* | 2<br>3<br>6 | 2<br>3<br>6* |
| | 2 | *A*<br>*B* | 1<br>2 | 1<br>2* |
| 1 | 0 | *A*<br>*C* | 2<br>3 | 2 + 3 = 5<br>3 + 6 = 9* |
| | 1 | *A*<br>*B*<br>*C* | 1<br>3<br>6 | 1 + 3 = 4<br>3 + 6 = 9*<br>6 + 2 = 8 |
| | 2 | *A*<br>*B* | 5<br>5 | 5 + 6 = 11*<br>5 + 2 = 7 |
| 0 | 0 | *A*<br>*B*<br>*C* | 4<br>3<br>5 | 4 + 9 = 13<br>3 + 9 = 12<br>5 + 11 = 16* |

**b** Maximum profit is £16 000
Tracing back through calculations the optimal strategy is *CAC*.

**6**

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| July 3 | 2 | 1 | 0 | 200 + 1250 = 1450* |
| | 1 | 2 | 0 | 100 + 1250 = 1350* |
| | 0 | 3 | 0 | 0 + 1250 = 1250* |
| June 5 | 2 | 5 | 2 | 200 + 1550 + 1450 = 3200 |
| | | 4 | 1 | 200 + 1250 + 1350 = 2800 |
| | | 3 | 0 | 200 + 1250 + 1250 =2700* |
| | 1 | 6 | 2 | 100 + 1550 + 1450 = 3100 |
| | | 5 | 1 | 100 + 1550 + 1350 = 3000 |
| | | 4 | 0 | 100 + 1250 + 1250 = 2600* |
| | 0 | 6 | 1 | 0 + 1550 + 1350 = 2900 |
| | | 5 | 0 | 0 + 1550 + 1250 = 2800* |
| May 7 | 2 | 6 | 1 | 200 + 1550 + 2600 = 4350* |
| | | 5 | 0 | 200 + 1550 + 2800 = 4550 |
| | 1 | 6 | 0 | 100 + 1550 + 2800 = 4450* |
| April 3 | 0 | 5 | 2 | 0 + 1550 + 4350 = 5900 |
| | 0 | 4 | 1 | 0 + 1250 + 4450 = 5700* |

The production schedule is:

| Month | April | May | June | July |
|---|---|---|---|---|
| Trailers | 4 | 6 | 5 | 3 |

Minimum cost = £5700

**7 a** Minimax
**b** *SAEHJT*
**c** *SCFHJT*

**Challenge**

**1** The shortest path from *A* to *F* is *ABECGF*.
The complication in trying to use dynamic programming here is that the stages are not defined.

**2** Let $x_1$, $x_2$, $x_3$, $x_4$ = the number of trailers built in April, May, June and July respectively.
Minimise

$$P = \sum_{i=1}^{4}\left(1250\left\{\frac{x_i}{6}\right\} + 300\left\{\frac{x_i}{5}\right\}\right) + 100(x_1 - 3) + 100(x_1 + x_2 - 10) + 100(x_1 + x_2 + x_3 - 15)$$

subject to
$0 \leqslant x_i \leqslant 6$
$3 \leqslant x_1 \leqslant 5$
$10 \leqslant x_1 + x_2 \leqslant 12$
$15 \leqslant x_1 + x_2 + x_3 \leqslant 17$
$x_1 + x_2 + x_3 + x_4 = 18$

## CHAPTER 6

### Prior knowledge check

**1** 0.3 × £0 + 0.3 × £0.50 + 0.3 × £1 + 0.1 × £5 = £0.95

**2**

| Basic variable | $x$ | $y$ | $z$ | $r$ | $s$ | Value |
|---|---|---|---|---|---|---|
| $r$ | 1 | 1 | 2 | 1 | 0 | 10 |
| $s$ | 2 | –3 | 0 | 0 | 1 | 12 |
| $P$ | –4 | 3 | –1 | 0 | 0 | 0 |

### Exercise 6A

**1 a** To be a zero-sum game, the amount won by *B* must be the negative of the amount won by *A* for every combination of choices. All of the pay-off values in the matrix are positive, so it cannot be a zero-sum game.
**b** *A* wins 1, *B* wins 2
**c** *A* plays 3, *B* plays 3

**2 a** The play-safe strategy for the two player is the choice that corresponds to the row maximin. The play-safe strategy for the column player is the choice that corresponds to the column minimax.
**b** For a zero-sum game: *A* wins 7, *B* loses 7
**c** Row maximin = 4, column minimax = 5
**d** The play-safe strategies are *A* plays 2, *B* plays 2
**e** The game does not have a stable solution since row maximin (4) ≠ column minimax (5)

**3 a** The play-safe strategies are *A* plays 1, *B* plays 2
**b** *A* plays 2

**4 a** *A* should play 1 (row maximin = 2)
*B* should play 2 (column minimax = 2)
**b** 2 is the smallest value in the row and largest value in the column. 2 = 2 (row maximin = column minimax) so game is stable and saddle point is (*A*1, *B*2).
**c** Value of the game for *A* is 2
**d**

| | *A* plays 1 | *A* plays 2 | *A* plays 3 |
|---|---|---|---|
| **B plays 1** | –3 | 2 | –4 |
| **B plays 2** | –2 | –1 | –2 |
| **B plays 3** | –3 | –3 | –1 |

Value of the game for *B* is –2.

**5 a** *R* should play 3 (row maximin = –1)
*S* should play 3 (column minimax = 1)
**b** Row maximin (–1) ≠ column minimax (1) so there is no stable solution.
**c**

| | Robert plays 1 | Robert plays 2 | Robert plays 3 |
|---|---|---|---|
| **Steve plays 1** | 2 | –2 | –1 |
| **Steve plays 2** | 1 | –3 | –1 |
| **Steve plays 3** | 3 | –1 | 1 |
| **Steve plays 4** | –1 | 2 | –3 |

**6 a** *A* should play 2 or 4 (row maximin = –1)
*B* should play 2 (column minimax = –1)
**b** Row maximin (–1) = column minimax (–1) so there is a stable solution. The saddle points are (*A*2, *B*2) and (*A*4, *B*2).
**c** Value of the game is –1 to *A*. (If *A* plays 2 or 4 and *B* plays 2 the value of the game is –1.)

**7 a** In a zero-sum game the two entries in each cell in the pay-off matrix add up to zero. A pay-off matrix for a zero-sum game uses only one value in each position. It is always written from the row player's (*A*'s) point of view unless you are told otherwise.
**b** *C* plays 2 (row maximin = –1)
*D* plays 3 (column minimax = 2)
**c** Row maximin (–1) ≠ column minimax (2) so there is no stable solution.
**d** If *C* plays 2 and *D* plays 3, the value of the game is 1 to Claire.
**e** *Either* Since the value of the game is 1 to Claire and it is a zero-sum game, the value of the game must be –1 to David.
*Or* If *C* plays 2 and *D* plays 3 Claire wins 1, so David wins –1

**f**

| | *C* plays 1 | *C* plays 2 | *C* plays 3 | *C* plays 4 |
|---|---|---|---|---|
| *D* plays 1 | −7 | −4 | 2 | −3 |
| *D* plays 2 | −2 | 1 | −5 | 3 |
| *D* plays 3 | 3 | −1 | −2 | 4 |
| *D* plays 4 | −5 | −3 | 1 | −2 |

**8 a** *H* plays 1 or 2
*D* plays 3 or 4

**b** Row maximin (0) = column minimax (0) so there is a stable solution. Saddlepoints (*H*1, *D*3) (*H*2, *D*3) (*H*1, *D*4) (*H*2, *D*4)

**c** The value of the game to Hilary = 0

**d** The value of the game to Denis = 0

**e**

| | *H* plays 1 | *H* plays 2 | *H* plays 3 | *H* plays 4 | *H* plays 5 |
|---|---|---|---|---|---|
| *D* plays 1 | −2 | −4 | −1 | −1 | 0 |
| *D* plays 2 | −1 | 0 | −4 | −1 | 2 |
| *D* plays 3 | 0 | 0 | 1 | 1 | 3 |
| *D* plays 4 | 0 | 0 | 1 | 2 | 3 |
| *D* plays 5 | −2 | −2 | −3 | 0 | 1 |

**9 a** For every outcome either Arjan pays Beth, or Beth pays Arjan, so one player's win is the other's loss.

**b**

| | *B* plays 1 | *B* plays 2 | *B* plays 3 |
|---|---|---|---|
| *A* plays 1 | −1 | 1 | 2 |
| *A* plays 2 | 1 | −2 | 1 |
| *A* plays 3 | 2 | 1 | −3 |

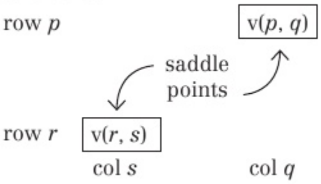
**c** Column minimax (1) ≠ row maximin (−1) so the solution is not stable.

**d**

| | *A* plays 1 | *A* plays 2 | *A* plays 3 |
|---|---|---|---|
| *B* plays 1 | 1 | −1 | −2 |
| *B* plays 2 | −1 | 2 | −1 |
| *B* plays 3 | −2 | −1 | 3 |

**10 a** $x \geqslant 3$ **b** Value = $\begin{cases} x & 3 \leqslant x \leqslant 5 \\ 5 & x \geqslant 5 \end{cases}$

**Challenge**




v(*r*, *s*) ⩾ v(*p*, *s*) since v(*r*, *s*) is the column maximum
v(*p*, *s*) ⩾ v(*p*, *q*) since v(*p*, *q*) is the row minimum.
Hence v(*r*, *s*) ⩾ v(*p*, *q*) (1)
v(*p*, *q*) ⩾ v(*r*, *q*) since v(*p*, *q*) is a column maximum
v(*r*, *q*) ⩾ v(*r*, *s*) since v(*r*, *s*) is a row minimum.
Hence v(*p*, *q*) ⩾ v(*r*, *s*) (2)
From (1) and (2): v(*p*, *q*) = v(*r*, *s*)

## Exercise 6B

**1** Row 3 dominates row 1, delete row 1.

| | Freya plays 1 | Freya plays 2 |
|---|---|---|
| Ellie plays 2 | −1 | 6 |
| Ellie plays 3 | 3 | −3 |

**2 a** Column 3 dominates column 2, delete column 2.

| | Harry plays 1 | Harry plays 3 |
|---|---|---|
| Doug plays 1 | −5 | −1 |
| Doug plays 2 | 2 | −6 |

**b**

| | Doug plays 1 | Doug plays 2 |
|---|---|---|
| Harry plays 1 | 5 | −2 |
| Harry plays 3 | 1 | 6 |

**3 a** Row 1 dominates row 2 and column 1 dominates column 2.

| | Nick plays 1 | Nick plays 2 |
|---|---|---|
| Chris plays 1 | 1 | 2 |
| Chris plays 3 | 2 | −1 |

**b**

| | Chris plays 1 | Chris plays 3 |
|---|---|---|
| Nick plays 1 | −1 | −2 |
| Nick plays 2 | −2 | 1 |

**4 a** *S*1 is always a better option than *S*2 so remove *S*2 and the matrix becomes:

| | Yin plays 1 | Yin plays 2 | Yin plays 3 |
|---|---|---|---|
| Sakiya plays 1 | 4 | −6 | 2 |
| Sakiya plays 3 | −5 | 7 | −8 |

**b** The row maximin is −6 for row 1. The column minimax is 2 for column 3. The best strategy for both players is (1, 3)

**c** There is no stable solution because row maximin (−6) ≠ column minimax (2)

**d** Play-safe value of game for Sakiya = 2

**e** Play-safe value of game for Yin = −2

**f**

| | Sakiya plays 1 | Sakiya plays 2 |
|---|---|---|
| Yin plays 1 | −4 | 5 |
| Yin plays 2 | 6 | −7 |
| Yin plays 3 | −2 | 8 |

**5 a** Each value in column 3 is ⩽ the corresponding values in column 2.

**b i**

| | Brian plays 1 | Brian plays 3 | Brian plays 4 |
|---|---|---|---|
| Ali plays 1 | 0 | 2 | 12 |
| Ali plays 2 | 9 | 8 | 10 |
| Ali plays 3 | 10 | 3 | 0 |

**ii** 8 is row minimum and column maximum so is a saddle point. Value of game to Brian is −8.

**c** The 8 in column 2 and row 2 is also a saddle point.

**Challenge**

Let the saddle point of *G* be in row *r* and column *s* of G.

**i** Suppose that dominance has reduced the number of rows of *G* by deleting row *p*.
(*r*, *s*) is the minimum value in row *r* of *G*.
(*r*, *s*) ⩾ (*p*, *s*) dominance argument
(*r*, *s*) is the maximum value in column *s*
Hence (*r*, *s*) is a saddle point of *G*

**ii** Suppose that dominance has reduced the number of columns of *G* by deleting column *q*.

$(r, s) \leqslant (r, q)$ dominance argument
so $(r, s)$ is the minimum value is row $r$ of $G$.
$(r, s)$ is the maximum value in column $s$
Hence $(r, s)$ is a saddle point of $G$.
From (i) and (ii), it follows that: Any stable solution of $G'$ must also be a stable solution of $G$.

## Exercise 6C

**1 a**

| Strategy | If *B* plays 1 | If *B* plays 2 | If *B* plays 3 |
|---|---|---|---|
| *V* | –1 | 4.5 | 0.5 |
| *W* | –0.8 | 4.6 | 0.2 |
| *X* | –0.6 | 4.7 | –0.1 |
| *Y* | –0.4 | 4.8 | –0.4 |
| *Z* | –0.2 | 4.9 | –0.7 |

**b** Strategy *Y*, with minimum pay-off –0.4

**2 a i** The row maximin (–1) ≠ column minimax (2) so there is no stable solution.
**ii** *A* should play 1 with probability $\frac{2}{5}$
*A* should play 2 with probability $\frac{3}{5}$
The value of the game to *A* is $3(\frac{2}{5}) - 1 = \frac{1}{5}$
**iii** *B* should play 1 with probability $\frac{7}{10}$
*B* should play 2 with probability $\frac{3}{10}$
The value of the game to *B* is $4(\frac{3}{10}) - 3 = -\frac{1}{5}$

**b i** The row maximin (–3) ≠ column minimax (2) so there is no stable solution.
**ii** *A* should play 1 with probability $\frac{3}{7}$
*A* should play 2 with probability $\frac{4}{7}$
The value of the game to *A* is $2 - 5(\frac{3}{7}) = -\frac{1}{7}$
**iii** *B* should play 1 with probability $\frac{9}{14}$
*B* should play 2 with probability $\frac{5}{14}$
The value of the game to *B* is $8(\frac{9}{14}) - 5 = \frac{1}{7}$

**c i** The row maximin (–1) ≠ column minimax (1) so there is no stable solution.
**ii** *A* should play 1 with probability $\frac{1}{3}$
*A* should play 2 with probability $\frac{2}{3}$
The value of the game to *A* is $7(\frac{1}{3}) - 2 = \frac{1}{3}$
**iii** *B* should play 1 with probability $\frac{2}{9}$
*B* should play 2 with probability $\frac{7}{9}$
The value of the game to *B* is $1 - 6(\frac{2}{9}) = -\frac{1}{3}$

**d i** The row maximin (–2) ≠ column minimax (1) so there is no stable solution.
**ii** *A* should play 1 with probability $\frac{3}{7}$
*A* should play 2 with probability $\frac{4}{7}$
The value of the game to *A* is $1 - 2(\frac{3}{7}) = \frac{1}{7}$
**iii** *B* should play 1 with probability $\frac{5}{7}$
*B* should play 2 with probability $\frac{2}{7}$
The value of the game to *B* is $4(\frac{5}{7}) - 3 = -\frac{1}{7}$

**3 a i** The row maximin (–4) ≠ column minimax (1) so there is no stable solution.
**ii** *A* should play 1 with probability $\frac{5}{12}$
*A* should play 2 with probability $\frac{7}{12}$
The value of the game to *A* is $1 - 6(\frac{5}{12}) = -\frac{3}{2}$

**b i** The row maximin (–2) ≠ column minimax (2) so there is no stable solution.
**ii** *A* should play 1 with probability $\frac{1}{2}$
*A* should play 2 with probability $\frac{1}{2}$
The value of the game to *A* is $3(\frac{1}{2}) - 1 = \frac{1}{2}$

**c i** The row maximin (–2) ≠ column minimax (3) so there is no stable solution.
**ii** *A* should play 1 with probability $\frac{9}{17}$
*A* should play 2 with probability $\frac{8}{17}$
The value of the game to *A* is $10(\frac{9}{17}) - 4 = \frac{22}{17}$

**d i** The row maximin (–3) ≠ column minimax (1) so there is no stable solution.
**ii** *A* should play 1 with probability $\frac{4}{11}$
*A* should play 2 with probability $\frac{7}{11}$
The value of the game to *A* is $1 - 3(\frac{4}{11}) = -\frac{1}{11}$

**4 a i** The row maximin (–1) ≠ column minimax (2) so there is no stable solution.
**ii** *B* should play 1 with probability $\frac{5}{9}$
*B* should play 2 with probability $\frac{4}{9}$
The value of the game to *B* is $2(\frac{5}{9}) - 1 = \frac{1}{9}$

**b i** The row maximin (–2) ≠ column minimax (3) so there is no stable solution.
**ii** *B* should play 1 with probability $\frac{7}{15}$
*B* should play 2 with probability $\frac{8}{15}$
The value of the game to *B* is $9(\frac{7}{15}) - 4 = \frac{3}{15}$

**c i** The row maximin (–2) ≠ column minimax (2) so there is no stable solution.
**ii** *B* should play 1 with probability $\frac{7}{13}$
*B* should play 2 with probability $\frac{6}{13}$
The value of the game to *B* is $5(\frac{7}{13}) - 2 = \frac{9}{13}$

**d i** The row maximin (–1) ≠ column minimax (2) so there is no stable solution.
**ii** *B* should play 1 with probability $\frac{5}{8}$
*B* should play 2 with probability $\frac{3}{8}$
The value of the game to *B* is $6(\frac{5}{8}) - 4 = -\frac{1}{4}$

**5 a** A zero-sum game is a game where the winnings of one player equal the losses of the other player.
**b** The row maximin (–2) ≠ column minimax (0) so there is no stable solution.
**c** *A* should play 1 with a probability of $\frac{5}{7}$ and play column 2 with a probability of $\frac{2}{7}$. The value of the game to *A* is $-\frac{4}{7}$.

**6 a** The row maximin (–2) ≠ the column minimax (1) so there is no stable solution.
**b** *B* should play 1 with a probability of $\frac{3}{5}$ and play 2 with a probability of $\frac{2}{5}$.
The value of the game to *B* is 1.
**c** The value of the game to *A* is –1.

**7 a**

| | *B* plays 1 | *B* plays 2 |
|---|---|---|
| *A* plays 1 | 2 | –3 |
| *A* plays 2 | –3 | 4 |

**b** Row maximin = –3, Column minimax = 2, so no stable solution
Optimal strategy for *A*: play 1 with probability $\frac{7}{12}$ and 2 with probability $\frac{5}{12}$
**c** Value to Amy = $-\frac{1}{12}$, so Amy can expect to lose about 8p every time they play. If Barun pays a fee of 10p then the game will be more fair, but will now be slightly biased in Amy's favour.

**8 a** The row maximin (1) ≠ the column minimax (4) so the game has no stable solution.
**b** Row *A*1 is always better than row *A*3 so *A*3 can be removed.

| | **B plays 1** | **B plays 2** | **B plays 3** |
|---|---|---|---|
| **A plays 1** | 6 | −2 | 2 |
| **A plays 2** | 1 | 4 | 5 |

**c** $A$ should play 1 with a probability of $\frac{3}{11}$ and play 2 with a probability of $\frac{8}{11}$. The value of the game to $A$ is $\frac{26}{11}$

**9 a** A saddle point in a pay-off matrix is a value which is the smallest in its row and the largest in its column.

**b** 4

**c** Column $B1$ is always better than column $B2$, so column $B2$ can be removed.

| | **B plays 1** | **B plays 3** |
|---|---|---|
| **A plays 1** | −4 | 3 |
| **A plays 2** | −1 | 0 |
| **A plays 3** | 1 | −4 |

**d** $B$ should play 1 with a probability of $\frac{2}{3}$ and play 3 with a probability of $\frac{1}{3}$.

**e** The value of the game to $B$ is $\frac{2}{3}$.

**10** Remove row 2
$A$ plays 1 with probability $\frac{6}{7}$
$A$ plays 3 with probability $\frac{1}{7}$
The value of the game to $A$ is $\frac{16}{7}$

**11** Bob plays 1 with probability $\frac{5}{8}$
Bob plays 2 with probability $\frac{3}{8}$
The value of the game to Bob is $\frac{1}{8}$
$\frac{1}{8} < \frac{1}{2}$ so Bob is incorrect.

**Challenge**

$$\text{Value} = \begin{cases} 0 & x < 0 \\ x & 0 \leqslant x \leqslant 1 \\ \dfrac{3x}{x+2} & x > 1 \end{cases}$$

## Exercise 6D

**1 a** Let $A$ play 1 with probability $p_1$
$A$ play 2 with probability $p_2$
$A$ play 3 with probability $p_3$
Let the value of the game to $A$ be $v$ and $V = v + 6$
Maximise $P = V$
subject to
$p_1 + 9p_2 + 7p_3 \geqslant V$
$\Rightarrow V - p_1 - 9p_2 - 7p_3 + r = 0$
$10p_1 + 3p_2 + 4p_3 \geqslant V$
$\Rightarrow V - 10p_1 - 3p_2 - 4p_3 + s = 0$
$7p_1 + 8p_2 + 5p_3 \geqslant V$
$\Rightarrow V - 7p_1 - 8p_2 - 5p_3 + t = 0$
$p_1 + p_2 + p_3 \leqslant 1$
$\Rightarrow p_1 + p_2 + p_3 + u = 1$
$p_1, p_2, p_3, r, s, t, u \geqslant 0$

**b** Let $A$ play 1 with probability $p_1$
$A$ play 2 with probability $p_2$
$A$ play 3 with probability $p_3$
Let the value of the game to $A$ be $v$ and $V = v + 5$
Maximise $P = V$
subject to
$2p_1 + 4p_2 + 7p_3 \geqslant V$
$\Rightarrow V - 2p_1 - 4p_2 - 7p_3 + r = 0$
$7p_1 + 3p_2 + p_3 \geqslant V$
$\Rightarrow V - 7p_1 - 3p_2 - p_3 + s = 0$
$4p_1 + 6p_2 + 3p_3 \geqslant V$
$\Rightarrow V - 4p_1 - 6p_2 - 3p_3 + t = 0$
$p_1 + p_2 + p_3 \leqslant 1$
$\Rightarrow p_1 + p_2 + p_3 + u = 1$
$p_1, p_2, p_3, r, s, t, u \geqslant 0$

**2** Let $A$ play 1 with probability $p_1$
$A$ play 2 with probability $p_2$
$A$ play 3 with probability $p_3$
Let the value of the game to $A$ be $v$ and $V = v + 4$
Maximise $P = V$
subject to
$6p_1 + 2p_2 + 5p_3 \geqslant V$
$\Rightarrow V - 6p_1 - 2p_2 - 5p_3 + r = 0$
$p_1 + 8p_2 + 3p_3 \geqslant V$
$\Rightarrow V - p_1 - 8p_2 - 3p_3 + s = 0$
$3p_1 + 5p_2 + 4p_3 \geqslant V$
$\Rightarrow V - 3p_1 - 5p_2 - 4p_3 + t = 0$
$p_1 + p_2 + p_3 \leqslant 1$
$\Rightarrow p_1 + p_2 + p_3 + u = 1$
$p_1, p_2, p_3, r, s, t, u \geqslant 0$

**3 a** Let $B$ play 1 with probability $q_1$
$B$ play 2 with probability $q_2$
$B$ play 3 with probability $q_3$
Let the value of the game to $B$ be $v$ and $V = v + 5$
Maximise $P = V$
subject to
$10q_1 + q_2 + 4q_3 \geqslant V$
$\Rightarrow V - 10q_1 - q_2 - 4q_3 + r = 0$
$2q_1 + 8q_2 + 3q_3 \geqslant V$
$\Rightarrow V - 2q_1 - 8q_2 - 3q_3 + s = 0$
$4q_1 + 7q_2 + 6q_3 \geqslant V$
$\Rightarrow V - 4q_1 - 7q_2 - 6q_3 + t = 0$
$q_1 + q_2 + q_3 \leqslant 1$
$\Rightarrow q_1 + q_2 + q_3 + u = 1$
$q_1, q_2, q_3, r, s, t, u \geqslant 0$

**b** Let $B$ play 1 with probability $q_1$
$B$ play 2 with probability $q_2$
$B$ play 3 with probability $q_3$
Let the value of the game to $B$ be $v$ and $V = v + 3$
Maximise $P = V$
subject to
$6q_1 + q_2 + 4q_3 \geqslant V$
$\Rightarrow V - 6q_1 - q_2 - 4q_3 + r = 0$
$4q_1 + 5q_2 + 2q_3 \geqslant V$
$\Rightarrow V - 4q_1 - 5q_2 - 2q_3 + s = 0$
$q_1 + 7q_2 + 5q_3 \geqslant V$
$\Rightarrow V - q_1 - 7q_2 - 5q_3 + t = 0$
$q_1 + q_2 + q_3 \leqslant 1$
$\Rightarrow q_1 + q_2 + q_3 + u = 1$
$q_1, q_2, q_3, r, s, t, u \geqslant 0$

**c** Let $B$ play 1 with probability $q_1$
$B$ play 2 with probability $q_2$
$B$ play 3 with probability $q_3$
Let the value of the game to $B$ be $v$ and $V = v + 5$
Maximise $P = V$
subject to
$3q_1 + 8q_2 + 6q_3 \geqslant V$
$\Rightarrow V - 3q_1 - 8q_2 - 6q_3 + r = 0$
$7q_1 + q_2 + 4q_3 \geqslant V$
$\Rightarrow V - 7q_1 - q_2 - 4q_3 + s = 0$
$4q_1 + 6q_2 + 5q_3 \geqslant V$
$\Rightarrow V - 4q_1 - 6q_2 - 5q_3 + t = 0$
$q_1 + q_2 + q_3 \leqslant 1$
$\Rightarrow q_1 + q_2 + q_3 + u = 1$
$q_1, q_2, q_3, r, s, t, u \geqslant 0$

**4 a** Let $A$ play 1 with probability $p_1$
$A$ play 2 with probability $p_2$
$A$ play 3 with probability $p_3$
Let the value of the game to $A$ be $v$ and $V = v + 5$
Maximise $P = V$
subject to
$4p_1 + 8p_2 + 3p_3 \geqslant V$
$\Rightarrow V - 4p_1 - 8p_2 - 3p_3 + r = 0$
$6p_1 + p_2 + 7p_3 \geqslant V$
$\Rightarrow V - 6p_1 - p_2 - 7p_3 + s = 0$
$p_1 + p_2 + p_3 \leqslant 1$
$\Rightarrow p_1 + p_2 + p_3 + t = 1$
$p_1, p_2, p_3, r, s, t \geqslant 0$

**b**

| b.v. | $V$ | $p_1$ | $p_2$ | $p_3$ | $r$ | $s$ | $t$ | Value |
|---|---|---|---|---|---|---|---|---|
| $r$ | ① | −4 | −8 | −3 | 1 | 0 | 0 | 0 |
| $s$ | 1 | −6 | −1 | −7 | 0 | 1 | 0 | 0 |
| $t$ | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $P$ | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**c** $\frac{44}{9} - 5 = -\frac{1}{9}$

**5 a** Let $B$ play 1 with probability $q_1$
$B$ play 2 with probability $q_2$
$B$ play 3 with probability $q_3$
Let the value of the game to $B$ be $v$ and $V = v + 4$
Maximise $P = V$
subject to $9q_1 + 2q_2 + q_3 \geqslant V$
$\Rightarrow V - 9q_1 - 2q_2 - q_3 + r = 0$
$3q_1 + 7q_2 + 8q_3 \geqslant V$
$\Rightarrow V - 3q_1 - 7q_2 - 8q_3 + s = 0$
$q_1 + q_2 + q_3 \leqslant 1$
$\Rightarrow q_1 + q_2 + q_3 + t = 1$
$q_1, q_2, q_3, r, s, t \geqslant 0$

**b**

| b.v. | $V$ | $p_1$ | $p_2$ | $p_3$ | $r$ | $s$ | $t$ | Value |
|---|---|---|---|---|---|---|---|---|
| $r$ | ① | −9 | −2 | −1 | 1 | 0 | 0 | 0 |
| $s$ | 1 | −3 | −7 | −8 | 0 | 1 | 0 | 0 |
| $t$ | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |
| $P$ | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**c** $\frac{69}{13} - 4 = \frac{17}{13}$

**6 a** Row maximin (2) ≠ column minimax (4)

**b**

| | B plays 1 | B plays 2 | B plays 3 |
|---|---|---|---|
| A plays 1 | 5 | 3 | −1 |
| A plays 2 | 4 | 5 | 2 |
| A plays 4 | 7 | −2 | 4 |

**c** Let $A$ play 1 with probability $p_1$
play 2 with probability $p_2$
play 4 with probability $p_4$
Adding 3 to each element of the pay-off matrix gives

| | B plays 1 | B plays 2 | B plays 3 |
|---|---|---|---|
| A plays 1 | 8 | 6 | 2 |
| A plays 2 | 7 | 8 | 5 |
| A plays 4 | 10 | 1 | 7 |

Let the value of the game to $A$ be $v$ and $V = v + 3$
The objective is to maximise $P = V$ subject to
$V \leqslant 8p_1 + 7p_2 + 10p_4$
$V \leqslant 6p_1 + 8p_2 + p_4$
$V \leqslant 2p_1 + 5p_2 + 7p_4$
$p_1 + p_2 + p_3 \leqslant 1$

## Mixed exercise 6

**1** $A$ should play 1 with probability $\frac{11}{17}$
$A$ should play 2 with probability $\frac{6}{17}$
The value of the game to $A$ is $\frac{14}{17}$
$B$ should play 1 with probability $\frac{8}{17}$
$B$ should play 2 with probability $\frac{9}{17}$
The value of the game to $B$ is $-\frac{14}{17}$

**2 a** No, the value of game to player $A$ can be non-zero. The sum of the values of the game to player $A$ and player $B$ is zero.

**b** The row maximin (−3) = the column minimax (−3) so the game has a stable solution.

**c** Mariette plays 2, Nigel plays 2. The value of the game to Mariette is −3.

**3 a** A play-safe strategy is one where each player looks for the worst that could happen and then picks the least worst option.

**b** $A$ plays 4, $B$ plays 2

**c** The row maximin (−1) = the column minimax (−1) so there is a stable solution to the game.

**d** The saddle point is the intersection of row 4 and column 2.

**e** The value of the game to $A$ at this point is −1.

**4 a** The row maximin (−2) ≠ the column minimax (4) so there is no stable solution to the game.

**b** Tadashi should play 1 with probability of 0.5 and play 2 with probability 0.5. His expected winnings are 1.

**5 a** The row maximin (−1) ≠ the column minimax (1) so there is no stable solution to the game.

**b** $A$ should play 1 with probability of 0.5 and play 2 with probability 0.5.

**c** The value of the game to $A$ is −0.5.

**6 a** In a pure strategy a player always makes the same choice. In a mixed strategy each action is assigned a probability, and is selected with the given probability.

**b** Olivia should play 1 with probability of $\frac{3}{7}$ and play 2 with probability $\frac{4}{7}$.

**c** The value of the game to Olivia is $-\frac{2}{7}$.

**7 a** Column 3 dominates column 2 (since $3 < 4$ and $-4 < -1$)

**b** $A$ should play 1 with probability $\frac{5}{13}$
$A$ should play 2 with probability $\frac{8}{13}$
The value of the game is $-\frac{17}{13}$
$B$ should play 1 with probability $\frac{7}{13}$
$B$ should play 2 with probability $\frac{6}{13}$
The value of the game is $\frac{17}{13}$

**8 a** $A$ plays 2 and $B$ plays 1.

**b** Row maximin (−4) ≠ column minimax (1) so there is no stable solution.

**c** Row 1 dominates row 3
(since $-5 > -7$  $2 > 0$  $3 > 1$)

| | G plays 1 | G plays 2 | G plays 3 |
|---|---|---|---|
| C plays 1 | −5 | 2 | 3 |
| C plays 2 | 1 | −3 | −4 |

**d** Cait should play 1 with probability $\frac{5}{13}$
Cait should play 2 with probability $\frac{8}{13}$
Cait should play 3 never
The value of the game is $-\frac{17}{13}$

**e** The value of the game to Georgi is $\frac{17}{13}$

**9 a** Row maximin (−2) ≠ column minimax (1) so there is no stable solution.

**b** A row $x$ dominates a row $y$ if in each column the element in row $x \geqslant$ the element in row $y$.

**c** Row 4 dominates row 3

| | B plays 1 | B plays 2 | B plays 3 |
|---|---|---|---|
| A plays 1 | 2 | −1 | −3 |
| A plays 2 | −2 | 1 | 4 |
| A plays 3 | −1 | 2 | −2 |

**d** Let $A$ play 1 with probability $p_1$
Let $A$ play 2 with probability $p_2$
Let $A$ play 3 with probability $p_3$
Let the value of the game to $A$ be $v$ so $V = v + 4$
Maximise $P = V$
subject to $6p_1 + 2p_2 + 3p_3 \geqslant V$
$3p_1 + 5p_2 + 6p_3 \geqslant V$
$p_1 + 8p_2 + 2p_3 \geqslant V$
$p_1 + p_2 + p_3 \leqslant 1$

**10 a** $A$ plays 1, $B$ plays 2
**b** Row maximin (–1) ≠ column minimax (2) so there is no stable solution.
**c**

| | B plays 2 | B plays 3 |
|---|---|---|
| **A plays 1** | –3 | 1 |
| **A plays 2** | –4 | 4 |
| **A plays 3** | 2 | –1 |

**d**

| | A plays 1 | A plays 2 | A plays 3 |
|---|---|---|---|
| **B plays 2** | 3 | 4 | –2 |
| **B plays 3** | –1 | –4 | 1 |

**e** $B$ should play 2 with probability $\frac{5}{11}$
$B$ should play 3 with probability $\frac{6}{11}$
The value of the game is $-\frac{4}{11}$

**11 a** $A$ plays 2, $B$ plays 1
**b** Row maximin (0) ≠ column minimax (5) so there is no stable solution.
**c**

| | A plays 1 | A plays 2 | A plays 3 |
|---|---|---|---|
| **B plays 1** | –2 | –5 | 2 |
| **B plays 2** | –7 | 0 | –3 |
| **B plays 3** | 1 | –8 | –5 |

**d** Let $B$ play 1 with probability $p_1$, play 2 with probability $p_2$ and play 3 with probability $p_3$
Let $v$ = value of the game to $B$ so $V = v + 9$
Maximise $P = V$
subject to $7p_1 + 2p_2 + 10p_3 \geqslant V$
$\Rightarrow V - 7p_1 - 2p_2 - 10p_3 + r = 0$
$4p_1 + 9p_2 + p_3 \geqslant V$
$\Rightarrow V - 4p_1 - 9p_2 - p_3 + s = 0$
$11p_1 + 6p_2 + 4p_3 \geqslant V$
$\Rightarrow V - 11p_1 - 6p_2 - 4p_3 + t = 0$
$p_1 + p_2 + p_3 \leqslant 1$
$\Rightarrow p_1 + p_2 + p_3 + u = 0$
where $p_1, p_2, p_3, r, s, t, u \geqslant 0$

**e**

| b.v. | $V$ | $p_1$ | $p_2$ | $p_3$ | $r$ | $s$ | $t$ | $u$ | Value |
|---|---|---|---|---|---|---|---|---|---|
| $r$ | 1 | –7 | –2 | –10 | 1 | 0 | 0 | 0 | 0 |
| $s$ | 1 | –4 | –9 | –1 | 0 | 1 | 0 | 0 | 0 |
| $t$ | 1 | –11 | –6 | –4 | 0 | 0 | 1 | 0 | 0 |
| $u$ | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| $P$ | –1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

**Challenge**

**a** For player $A$, play 1 with probability $\frac{d-c}{a-b-c+d}$ and 2 with probability $\frac{a-b}{a-b-c+d}$

For player $B$, play 1 with probability $\frac{d-b}{a-b-c+d}$ and 2 with probability $\frac{a-c}{a-b-c+d}$

**b** Value to $A = \frac{ad-bc}{a-b-c+d}$, Value to $B = \frac{bc-ad}{a-b-c+d}$
**c** Hence value to $B$ is the negative of value to $A$.

## CHAPTER 7

### Prior knowledge check

**1 a** 2, 6, 18, 54, 162 **b** $2(3^n)$
**2** $a = 2$, $b = -1$

### Exercise 7A

**1 a** $u_n = 1.05u_{n-1}$, $u_0 = 7000$
**b** £8508.54
**2 a** $d_n = 0.78d_{n-1} + 25$, $d_0 = 156$
**b** 134 ml
**3** Each month 5% is added to the balance so
Balance + interest = $b_{n-1} + 0.005b_{n-1} = 1.005b_{n-1}$
£200 is paid off so this amount is reduced by £200.
$k = 1.005$.
**4** $P_n = 1.01P_{n-1} + 50\,000$, $P_0 = 12\,500\,000$
**5** $u_{n-1} = 5n - 3$, so $u_{n-1} + 5 = 5n + 2 = u_n$
**6** $u_{n-1} = 6 \times 2^{n-1} + 1$, so $2u_{n-1} - 1 = 6 \times 2^n + 1 = u_n$
**7 a** 1, 4, 9, 16
**b** $u_{n+1} = \sum_{i=1}^{n}(2i - 1) + (2(n + 1) - 1) = u_n + 2n + 1$, $n \geqslant 1$
**c** $u_{n+1} = (n + 1)^2 = n^2 + 2n + 1 = u_n + 2n + 1$
**8 a i** $2000 \times 1.01^{n-1}$ **ii** $1780 + 20n$
**b** $s_n = s_{n-1} + 2000 \times 1.01^{n-1} - 1780 - 20n$
**9 a** With 1 person there are no handshakes.
**b** $h(n + 1) = h(n) + n$
**10 a** 1, 1, 5, 13, 41, 121
**b** 1, 1, –1, –3, –1, 5
**c** 1, 1, 6, 13, 27, 50
**11** $B_n = 2B_{n-1} - B_{n-2}$, $n \geqslant 2$; $B_0 = 100$
**12** $u_{n-1} = (3 - n)2^n$, $u_{n-2} = (4 - n)2^{n-1}$
$4(u_{n-1} - u_{n-2}) = (3 - n)2^{n+2} - (4 - n)^{2n+1}$
$= (6 - 2n - 4 + n)2^{n+1} = (2 - n)2^{n+1} = u_n$
**13 a** 10, 10, 10, 10; 20, 10 , 10; 10, 20, 10; 10, 10, 20; 20, 20. $J_4 = 5$
**b** $J_n = J_{n-1} + J_{n-2}$, $J_1 = 1$, $J_2 = 2$
**c** 34 ways
**14 a** e.g. Initially there are 4 rabbits so $F_0 = 4$.
$F_1 = 6 \times 4 + 4 = 28$. Each subsequent year the $F_{n-1} - F_{n-2}$ rabbits just born produce 2 offspring each, and the $F_{n-2}$ older rabbits produce 6 offspring.
So $F_n = 2(F_{n-1} - F_{n-2}) + 6F_{n-2} + F_{n-1} = 3F_{n-1} + 4F_{n-2}$ as required.
**b** e.g. assumes no female rabbits ever die.
**15 a** $b_1 = 2$, $b_3 = 3$
**b** Strings of length $n$ ending with 0 that do not have consecutive 1s are the strings of length $n - 1$ with no consecutive 1s with a 0 at the end, so there are $b_{n-1}$ such strings.
Bit strings of length $n$ ending with 1 that do not have consecutive 1s must have 0 as their $(n - 1)$ th digit; otherwise they will end with a pair of 1s. It follows that the strings with length n ending with a 1 that have no consecutive 1s are the strings of length $n - 2$ with no consecutive 1s with 01 added at the end, so there are $b_{n-2}$ such strings.
We conclude that $b_n = b_{n-1} + b_{n-2}$.
**c** $b_7 = 34$

### Exercise 7B

**1 a** $u_n = 5(2^n)$ **b** $b_n = 4\left(\frac{5}{2}\right)^{n-1}$
**c** $d_n = 10\left(-\frac{11}{10}\right)^{n-1}$ **d** $x_n = 2(-3)^n$

**2 a** $u_n = 5 + 3n$ **b** $x_n = 2 + \frac{1}{2}n + \frac{1}{2}n^2$
**c** $y_n = 3 - 2n + \frac{1}{6}n(n+1)(2n+1)$
**d** $s_n = 1 - 2n + n^2$

**3 a** $a_n = 2^n - 1$ **b** $u_n = 2(-1)^{n-1} + 1$
**c** $h_n = \frac{1}{2}(7 \times 3^n - 5)$ **d** $b_n = 2 + (-2)^{n-1}$

**4 a** $n - 1$ teams play each other $g_{n-1}$ times. When an $n$th team is added, this team has to play each of the other $n - 1$ teams once, so there are $g_{n-1} + n - 1$ games in total. i.e. $g_n = g_{n-1} + n - 1$. $g_1 = 0$.
**b** $g_n = g_1 + \sum_{r=2}^{n} r - \sum_{r=2}^{n} 1$
$= 0 + \frac{n(n+1)}{2} - 1 - (n-1) = \frac{n(n-1)}{2}$

**5 a** $u_n = c(4^n) + \frac{1}{3}$
**b i** $\frac{1}{3}(2 \times 4^n + 1)$
**ii** $\frac{1}{3}(1 - 4^{n-1})$
**iii** $\frac{1}{3}(599 \times 4^{n-1} + 1)$

**6 a** $u_n = c(3^n) - \frac{1}{2}n - \frac{3}{4}$
**b** $u_n = \frac{1}{4}(25 \times 3^{n-1} - 2n - 3)$

**7 a** $u_3 = 9.352$
**b** $u_n = 10 - 3(0.6^n)$
**c** $n = 7$

**8 a** $D_n = 0.95D_{n-1} + 20$, $D_0 = 200$
**b** $D_n = 200(2 - 0.95^n)$
**c** As $n \to \infty$, $0.95n \to 0$, so the deer population approaches 400 in the long term.

**9** $u_n = 6(4^n) + 1$

**10** $u_n = 2^{n+1} + 1$

**11** $u_n = \frac{1}{9}(71 \times 4^n - 6n - 8)$

**12 a** $u_n = -5(2^n - 201)$ **b** $u_8 = -275$

**13 a** $u_n = 2^n(c - n)$ **b** $2^n(\frac{5}{2} - n)$

**14 a** $u_1 = 1$, $u_2 = k + 1$, $u_3 = k^2 + k + 1$
**b** $u_n = \frac{k^n - 1}{k - 1}$
**c i** Tends to infinity
**ii** Tends to $\frac{1}{1-k}$
**iii** Alternates between 0 and 1
**iv** Diverges to $\pm\infty$ and alternates

**15 a** $3n^2 + 4n$ **b** $a_n = 3n^2 + 4n + 2$
**c** $n = 13$

**16 a** $u_n = 89 - n(n+1)(2n+1)$ **b** $u_4 = -91$
**c** Adding an odd number, 89, to an even number $n(n+1)(2n+1)$ gives an odd number.

**17 a** $u_n = 3 - n(n+1)$
**b** $-103 = 3 - n(n+1) \Rightarrow n(n+1) = 106 = 2 \times 53$, $n$ and $n+1$ are consecutive integers while 2 and 53 are not.
**c** $k = 21$

**18 a** $u_n = 1.015u_{n-1} - P$, $u_0 = 2000$
**b** $u_n = \frac{200}{3}(1.015^n(30 - P) + P)$
**c** $P = 127.61$

**Challenge**

**a** Disk cannot be moved from $A$ to $C$ in one jump, so must move from $A$ to $B$, then $B$ to $C$.
**b** $A \to B, B \to C, A \to B, C \to B, B \to A, B \to C, A \to B, B \to C$
**c** Transfer $n - 1$ disks from $A$ to $C$ ($H_{n-1}$ moves), then move $n$th disk from $A$ to $B$ (1 move), then transfer $n - 1$ disks from $C$ to $A$ ($H_{n-1}$ moves), then move $n$th disk from $B$ to $C$ (1 move), then transfer $n - 1$ disks from $A$ to $C$ ($H_{n-1}$ moves). In total, $H_n = 3H_{n-1} + 2$.
**d i** $H_n = 3^n - 1$ **ii** 59048 moves

## Exercise 7C

**1 a** $n$ even: $5(-1)^{n-1} + 6(-1)^{n-2} = -5 + 6 = 1 = (-1)^n$
$n$ odd: $5(-1)^{n-1} + 6(-1)^{n-2} = 5 - 6 = -1 = (-1)^n$
**b** $5 \times 6^{n-1} + 6 \times 6^{n-2} = 6^{n-1}(5 + 1) = 6^n$
**c** $5(A(-1)^{n-1} + B(6^{n-1})) + 6(A(-1)^{n-2} + B(6^{n-2}))$
$= -5A(-1)^{n-2} + 6A(-1)^{n-2} + 5B(6^{n-1}) + 6B(6^{n-2})$
$= A(-1)^{n-2} + 6B(6^{n-1}) = A(-1)^{n-2} + B(6^n)$

**2 a** $5(3^n) - 6 \times 5(3^{n-1}) + 9 \times 5(3^{n-2}) = (45 - 90 + 45)3^{n-2} = 0$
**b** $-n3^n - 6(-(n-1)3^{n-1}) + 9(-(n-2)3^{n-2})$
$= -n3^n + (6n - 6)3^{n-1} + (18 - 9n)3^{n-2}$
$= (-9n + 18n - 18 + 18 - 9n)3^{n-2} = 0$
**c** Follows from parts **a** and **b**.

**3 a** $\cos\left((n+2)\frac{\pi}{2}\right) + \cos\left(n\frac{\pi}{2}\right) = \cos\left(\pi + n\frac{\pi}{2}\right) + \cos\left(n\frac{\pi}{2}\right)$
$= -\cos\left(n\frac{\pi}{2}\right) + \cos\left(n\frac{\pi}{2}\right) = 0$
**b** $\sin\left((n+2)\frac{\pi}{2}\right) + \sin\left(n\frac{\pi}{2}\right) = \sin\left(\pi + n\frac{\pi}{2}\right) + \sin\left(n\frac{\pi}{2}\right)$
$= -\sin\left(n\frac{\pi}{2}\right) + \sin\left(n\frac{\pi}{2}\right) = 0$
**c** Follows from parts **a** and **b**.

**4** $au_{n-1} + bu_{n-2}$
$= a(c\mathrm{F}(n-1) + d\mathrm{G}(n-1)) + b(c\mathrm{F}(n-2) + d\mathrm{G}(n-2))$
$= c(a\mathrm{F}(n-1) + b\mathrm{F}(n-2)) + d(a\mathrm{G}(n-1) + b\mathrm{G}(n-2))$
$= c\mathrm{F}(n) + d\mathrm{G}(n) = u_n$

**5 a** $a_n = A + Bn$ **b** $u_n = A + B(2^n)$
**c** $x_n = (A + Bn)3^n$ **d** $t_n = A(2 + \mathrm{i})^n + B(2 - \mathrm{i})^n$

**6** $a = -8$, $b = 7$

**7 a** $a_n = 2^n + 3^n$ **b** $u_n = (7 - n)3^{n-2}$
**c** $s_n = 2^n + 3(5^n)$
**d** $u_n = \frac{5}{2}((1 + 2\mathrm{i})^{n-1} + (1 - 2\mathrm{i})^{n-1})$

**8 a** $u_n = \frac{61}{3} - \frac{1}{3}(4^n)$
**b** $u_{n+1} - u_n = -4^n < 0 \Rightarrow u_n$ is decreasing
$u_n < 0 \Rightarrow 4^n > 61 \Rightarrow n \geqslant 3$

**9 a** $u_n = 2^n\left(\cos\left(n\frac{\pi}{4}\right) + \left(\frac{\sqrt{2}}{2} - 1\right)\sin\left(n\frac{\pi}{4}\right)\right)$
**b** cos and sin are periodic of period $2\pi$, so period for $u_n$ is $\frac{2\pi}{\frac{\pi}{4}} = 8$.

**10 a** $L_1 = 1, L_2 = 3, L_3 = 4, L_4 = 7, L_5 = 11, L_6 = 18, L_7 = 29$
**b** Auxiliary equation is $r^2 - r - 1 = 0$, so $r = \frac{1 \pm \sqrt{5}}{2}$
and $L_n = A\left(\frac{1 + \sqrt{5}}{2}\right)^n + B\left(\frac{1 - \sqrt{5}}{2}\right)^n$
$L_1 = \frac{1}{2}A + \frac{\sqrt{5}}{2}A + \frac{1}{2}B - \frac{\sqrt{5}}{2}B = 1$
$L_2 = \frac{3}{2}A + \frac{\sqrt{5}}{2}A + \frac{3}{2}B - \frac{\sqrt{5}}{2}B = 3$
Solving these equations gives $A = B = 1$,
so the closed form is $L_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n$

**11 a** $x_n = A(2^n) + B(3^n) + \frac{1}{2}$
**b** $u_n = A(2^n) + B(-1)^n - n - \frac{5}{2}$
**c** $a_n = A(-3)^n + B(-1)^n - 5(-2)^n$
**d** $a_n = A(-3)^n + B(-1)^n + 2n(-3)^n$
**e** $a_n = \left(A + Bn + \frac{n^2}{18}\right)3^n$
**f** $u_n = A(2^n) + B(5^n) + 8 + 2n$

**12 a** $u_n = \frac{1}{8}(7(3^n) - 5(-1)^n - 2)$
**b** $a_n = 15 - 2^{n+1} + (-1)^{n+1}$
**c** $u_n = n5^{n+1} - 2 \times 5^n + 6(-2)^n$
**d** $x_n = (4n^2 - 8n + 6)5^n$

**13 a** $k = \frac{7}{9}$ **b** $b_n = \frac{2}{9}(-2)^n - \frac{5}{6}n(-2)^n + \frac{7}{9}$

**14 a** $u_n = A(6^n) + B - 15n$ **b** $u_n = 3(6^n) - 15n - 1$

**15 a** $k = \frac{7}{18}$ **b** $u_n = A(3^n) + Bn(3^n)$

**c** $u_n = \left(1 - \frac{1}{18}n + \frac{7}{18}n^2\right)3^n$

**16 a** Auxiliary equation is $r^2 - r + 1 = 0$, so $r = e^{\pm\frac{\pi}{3}}$, and the general solution has the form
$u_n = A\cos\left(\frac{\pi}{3}n\right) + B\sin\left(\frac{\pi}{3}n\right)$.
$u_0 = 0 \Rightarrow A = 0$, and $u_1 = 3 \Rightarrow \frac{\sqrt{3}}{2}B = 3 \Rightarrow B = 2\sqrt{3}$.
So the particular solution is $u_n = 2\sqrt{3}\sin\left(\frac{\pi}{3}n\right)$.

**b** $\sin\left(\frac{n\pi}{3}\right) = \sin\left(\frac{n\pi}{3} + 2\pi\right) = \sin\left(\frac{(n+6)\pi}{3}\right)$ so the sequence is periodic with period 6.

**17 a** 22

**b** $s_n = 2s_{n-1} + 2s_{n-2}$, $s_0 = 1$, $s_1 = 3$

**c i** $s_n = \frac{1}{6}\left((3 + 2\sqrt{3})(1 + \sqrt{3})^n + (3 - 2\sqrt{3})(1 - \sqrt{3})^n\right)$

**ii** 578 272 256

**Challenge**

**1** $u_n = 2^{\left(\frac{1}{2}\right)^n + 2(-1)^n}$

**2** $a = \sqrt{3}$, $b = -1$, $\max(u_n) = 2k$, $\min(u_n) = -2k$

## Mixed exercise 7

**1** $u_n = 3(2^n) + 1$

**2 a** $u_n = 2000 - \frac{1}{2}n(n + 1)$ **b** $u_{63} = -16$

**3 a** $u_n = \frac{5}{2}(3^n - 1)$ **b** 147 620

**c** $u_{14} = 11\,957\,420$

**4 a** $T_0$ = number of trees planted in first year = 12 000
Removing 20% of the trees compared to year $n - 1$ leaves 80% of this number of trees, i.e. $0.8T_{n-1}$, then to represent the 1000 trees planted, add 1000 to this to get $T_n = 0.8T_{n-1} + 1000$

**b** $T_n = 7000(0.8)^n + 5000$ **c** 5000

**5 a** $b_n = 1.0025b_{n-1} - 1200$, $b_0 = 175\,000$ **b** 2033

**6 a** $P_4 = 6$

**b** $P_n = P_1 + \sum_{r=2}^{n}(r - 1) = 0 + \sum_{r=2}^{n} r - (n - 1)$
$= \frac{1}{2}n(n + 1) - 1 - n + 1 = \frac{1}{2}n(n - 1)$

**c** 4950

**7 a** $t_5 = 25$, $t_6 = 36$, $t_7 = 49$

**b** $t_n = t_{n-1} + 2n - 1$

**c** $t_n = n^2$, $t_{100} = 10\,000$

**8 a** $\begin{pmatrix} 1 & 4 + 3p \\ 0 & 3q \end{pmatrix}$

**b** $a_n = 3a_{n-1} + 4$, $b_n = 3b_{n-1}$

**c** $a_n = 2(3^n - 1)$, $b_n = 3^n \Rightarrow \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & 2(3^n - 1) \\ 0 & 3^n \end{pmatrix}$

**9 a** $S_5 = 55$, $S_6 = 91$, $S_7 = 140$ **b** $S_n = S_{n-1} + n^2$

**c** $S_n = \frac{1}{6}n(n + 1)(2n + 1)$

**10 a** $u_n = 1.2u_{n-1} - k(2^n)$, $u_0 = 100$

**b** C.F. is $A(1.2^n)$ and P.S. is $-\frac{5k}{2}(2^n)$,
so $u_n = A(1.2n) - \frac{5k}{2}(2^n)$
Using $u_0 = 100$, $A = 100 + \frac{5k}{2}$, and hence
$u_n = \left(100 + \frac{5k}{2}\right)(1.2^n) - \frac{5k}{2}(2^n)$

**11 a** 

**b** There are $f_{n-1}$ paths of length $n$ ending in a small flagstone and $f_{n-2}$ paths of length $n$ ending in a long flagstone. This gives a total of $f_n = f_{n-1} + f_{n-2}$ paths of length $n$. There is one path of length 1 m and there are two paths of length 2 m, so $f_1 = 1$ and $f_2 = 2$.

**c** Solving the recurrence relation gives
$u_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{n+1}\right)$
So for $n = 200$, $u_{200} = \frac{1}{\sqrt{5}}\left(\left(\frac{1 + \sqrt{5}}{2}\right)^{201} - \left(\frac{1 - \sqrt{5}}{2}\right)^{201}\right)$

**12 a** $t_2 = 8$, $t_3 = 22$

**b** If the final digit of the string is not 0, then there are $t_{n-1}$ possibilities for the rest of the string for each of final digits 1 and 2. If the final digit is zero, then the penultimate digit must *not* be zero, i.e. can be either 1 or 2, and then there are $t_{n-2}$ possibilities for the rest of the string for each of these two cases. Thus, $t_n = 2t_{n-1} + 2t_{n-2}$

**c** $t_6 = 448$

**d i** $t_n = \frac{1}{2\sqrt{3}}\left((2+\sqrt{3})(1 + \sqrt{3})^n + (\sqrt{3} - 2)(1 - \sqrt{3})^n\right)$

**ii** 3 799 168

**13 a** $u_n = A(2^n) + B(-1)^n$ **b** $u_n = 2^{n-1}$

**14 a** $x_n = A(2^n) + B(5^n) + \frac{3}{4}$ **b** $x_n = \frac{1}{4}(5^{n-1} + 3)$

**15** $a_n = \frac{1}{60}(19(5^n) - 19(-3)^n - 2^{n+4})$

**16 a** $u_n = \cos\left(\frac{n\pi}{4}\right) + (\sqrt{2} - 1)\sin\left(\frac{n\pi}{4}\right)$

**b** cos and sin are periodic of period $2\pi$, so period for $u_n$ is $\frac{2\pi}{\frac{\pi}{4}} = 8$.

**17 a** $S_{n+2} = S_{n+1} + S_n$, $S_1 = 1$, $S_2 = 2$

**b** $S_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2}\right)^{n+1}\right)$

**Challenge**

**1 a** 31

**b** Consider one of the points already on the circumference of the circle ($A$), and one of the existing points of intersection of two diagonals ($B$). The line through these two points will meet the circle at two points, $A$ and $C$. Since there are finitely many pairs $\{A, B\}$, there will be finitely many points $C$ on the circumference of the circle such that the line $AC$ goes through an existing intersection point. However, since there are infinitely many points on the circumference of the circle, it is possible to choose one, $D$, which doesn't coincide with any of the points $C$, and thus the chords $AD$ do not go through any of the existing intersection points.

**c** $C_n = C_{n-1} + \frac{1}{6}n^3 - n^2 + \frac{17}{6}n - 2$

**d** $C_n = \frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1$; $C_{100} = 3\,926\,176$

**2 a** Walks of length 1 start at $A$ and end at any of the other points; the spider cannot return to $A$.

**b** 6

**c** $u_n = \frac{1}{4}(3^n + 3(-1)^n)$

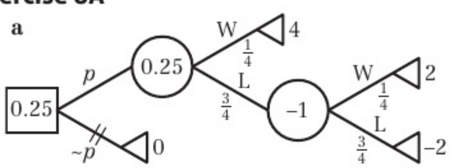
## CHAPTER 8

## Prior knowledge check

**1** 3.5

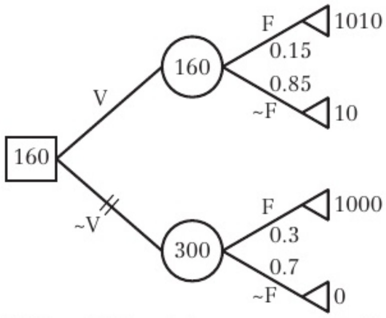
**2** 720

## Exercise 8A

**1 a** 

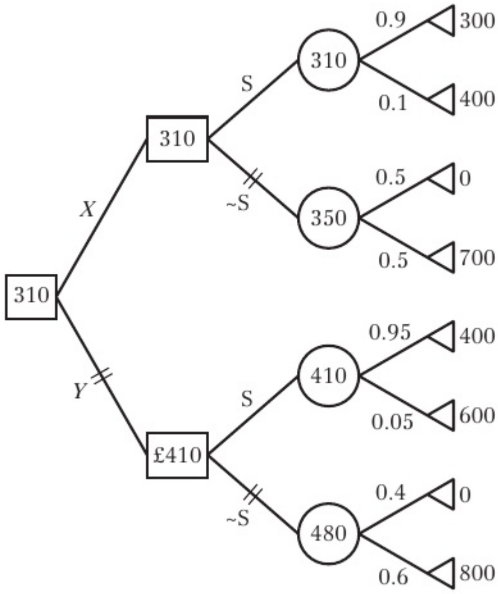

**b** The EMV for the game is £0.25 > £0 so Claire should play the game. If she doesn't win at the first attempt, then she should have a second go since −£1 > −£2

**2 a**



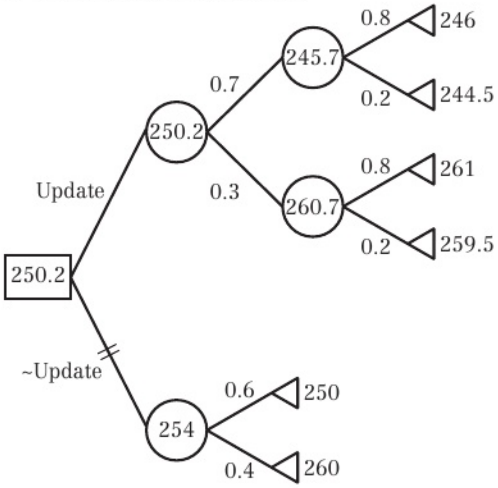

**b** £160 < £300 so it is more cost-effective for Stephen to have the vaccine.

**3 a**



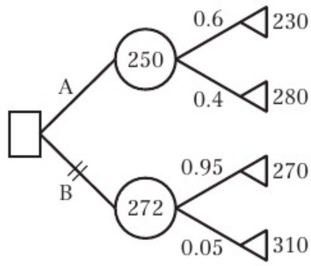

**b** Jemima's best strategy is to buy car *X* and pay for it to be serviced since 310 < 410.

**4 a** Amounts shown in £1000s



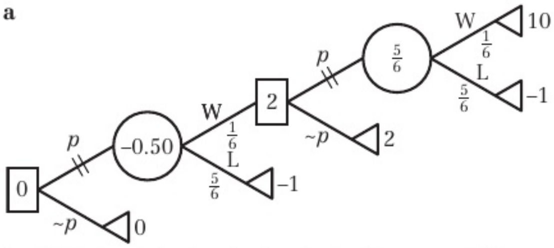

**b** The best strategy using the EMVs is not to update the kitchen since 250.2 < 254.

**5 a** Amounts shown in £1000s



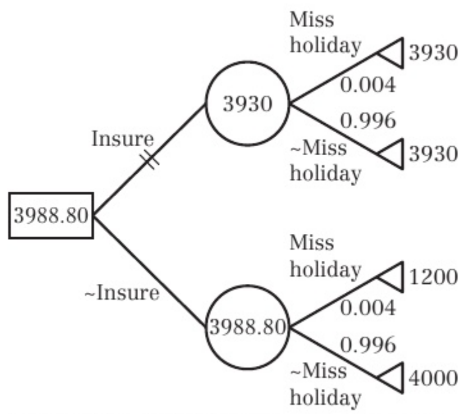

**b** Using the EMVs, the contract should be awarded to contractor *A* since 250 < 272.

**6 a**




**b** Beth's best strategy is to not play the game. If she chooses to play and wins on the first throw, then she should keep the winnings and not take the second throw.

**7 a**

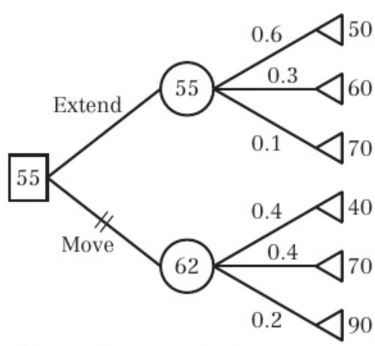



Using EMVs, the couple should not take out insurance since 3988.8 < 3930.

**b** The EMV associated with taking out insurance increases to 3990, which now makes it better to pay for the insurance cover.

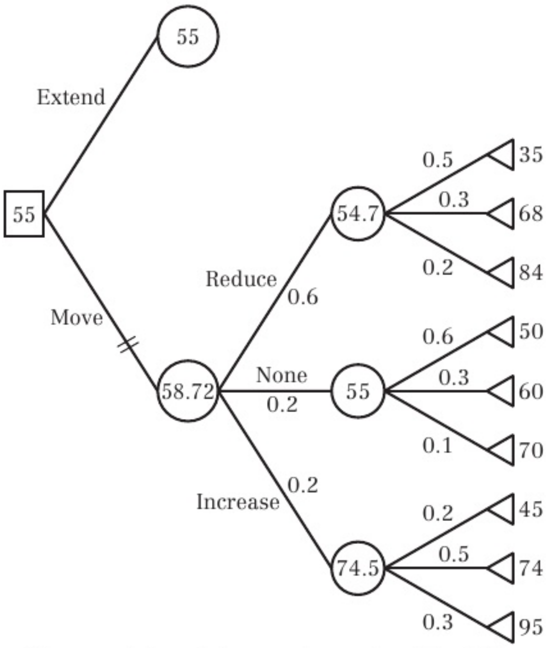
**c** £11.20

**8 a**




55 000 < 62 000 so the better option is to extend.

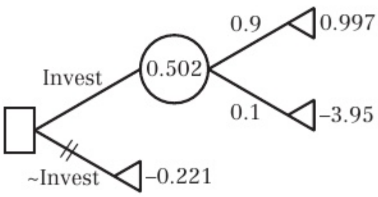
**b**




The expected costs to move has reduced, but it is still cheaper to extend.

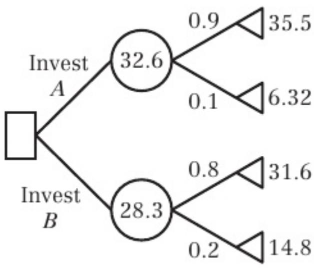
## Exercise 8B

1



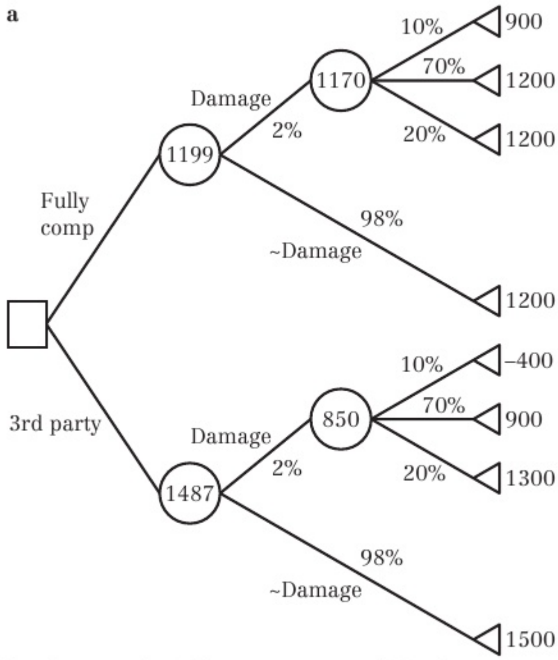

The investment should be made.

2 a




b The expected utilities are 32.6 utils (invest in *A*) and 28.3 utils (invest in *B*). The better option is to invest in *A*.

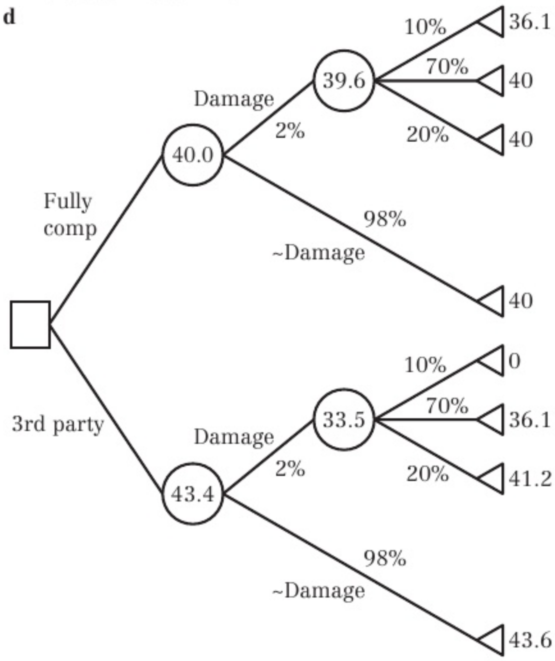
3 a




b The EMV for 3rd party insurance (£1487) is more than the EMV for fully comprehensive insurance (£1199.40). This suggests that 3rd party insurance is the better option.

c The EMV represents the average or long-term view. However, there is a risk that Sally could lose her car and may not be able to replace it and this is not reflected in the EMV.

d




With the given utility function, the expected utility values still favour 3rd party insurance.

4 $\frac{1}{3} \times \sqrt[3]{(x+2)^2} > \sqrt[3]{4}$

Minimum value of $x$ is 8.39 (2 d.p.)

**Challenge**

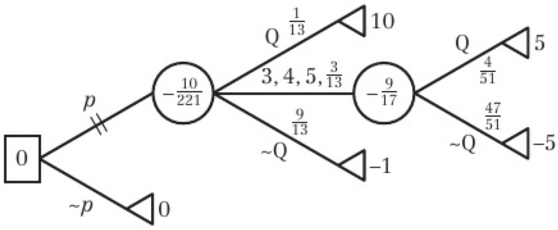
**a** EMV = 0.379 so Daniel should play

**b** $u(x) = \begin{cases} 2x & 0 \leqslant x \leqslant 50000 \\ 50000 + x & 50000 \leqslant x \leqslant 1000000 \\ 1000000 & x \geqslant 1000000 \end{cases}$

**c** Expected utility = –1.34 (3 s.f.) so James should not play

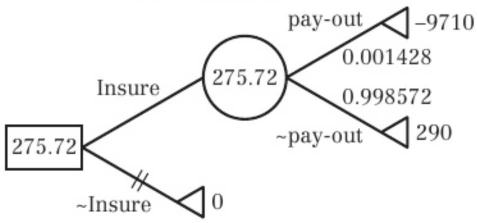
## Mixed exercise 8

**1**




Playing the game has an EMV of $-\frac{10}{221} < 0$ which is the EMV of not playing. Using the EMVs, Jacob should not play the game.

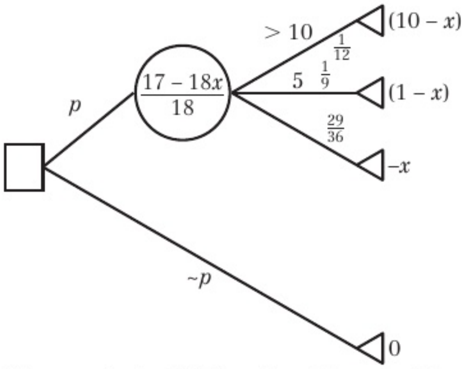
**2 a** $\text{P(pay-out)} = 1 - \left(1 - \frac{1}{67}\right)^{400}$
$= 0.01428$ (4 s.f.)




The EMV is £275.75 so the company should offer the cover.

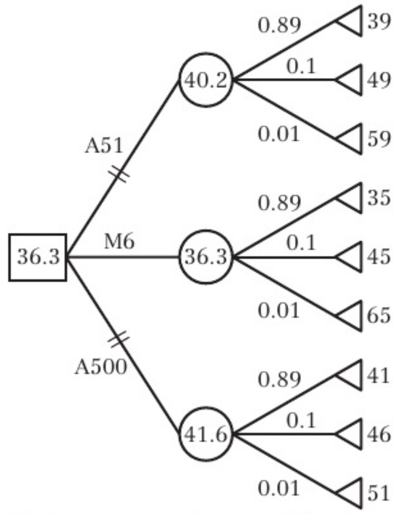
**b** The EMV criterion is appropriate here because it represents the long-term return to the company, which will be able to deal with the occasional loss.

**3**




When $x = 1$, the EMV is < 0, so Liam should not play the game.

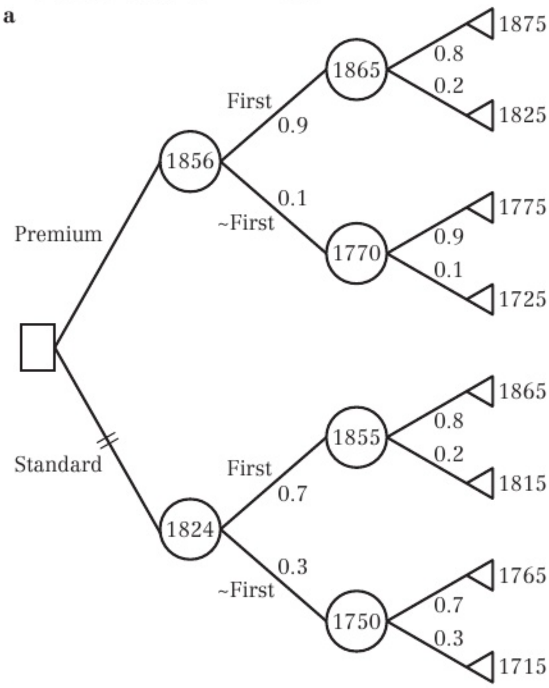
**4 a**




**b** Minimum expected time = 36.3 minutes on the M6.

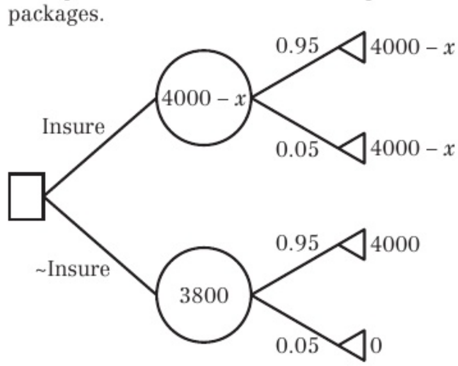
**c** If it is important to arrive on time, then it may be better to choose a route where the maximum expected delay is minimised.

**5 a**




**b** The optimum EMV is £1856 for the premium packages.

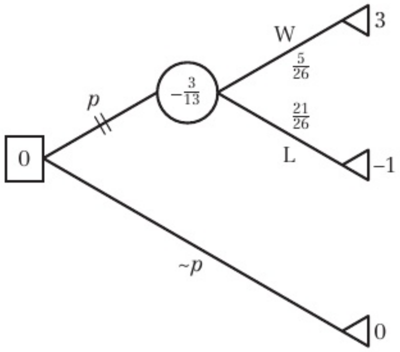
**6 a**



**b** £200

**c** $\sqrt[3]{4000 - x} > 0.95 \times \sqrt[3]{4000}$;
Revised figure = £570.50

**7 a**




Joe should not play the game.

**b** $\frac{5}{26}\sqrt[3]{(x + 1)^2} > 1$, £10.86

**8 a** The purpose of a utility function, in decision analysis, is to provide a customisable way to compare the value of outcomes taking into account, for example, the degree of aversion to risk.

**b** $p = 3810$

**Challenge**

**a** The utility function is designed to prevent the possibility of a very high profit having too much influence on decisions.

**b** Project A, £49 000

## Review exercise 2

**1** *SBEIT*, length 107

**2 a**

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| 1 | *D* | *DT* | *T* | 8* |
| | *E* | *ET* | *T* | 10* |
| | *F* | *FT* | *T* | 6* |
| 2 | *A* | *AD* | *D* | Max (7, 8) = 8* |
| | | *AE* | *E* | Max (8, 10) = 10 |
| | *B* | *BE* | *E* | Max (9, 10) = 10 |
| | | *BF* | *F* | Max (3, 6) = 6* |
| | *C* | *CE* | *E* | Max (6, 10) = 10 |
| | | *CF* | *F* | Max (9, 6) = 9* |
| 3 | *S* | *SA* | *A* | Max (9, 8) = 9 |
| | | *SB* | *B* | Max (7, 6) = 7* |
| | | *SC* | *C* | Max (6, 9) = 9 |

**b** Minimax route is *SBFT*.

**c** Minimum fuel capacity is 7 tonnes.

**d** For example, there may be delays landing, or the aircraft may be diverted, which would require additional fuel.

**3 a** The route from start to finish in which the arc of minimum length is as large as possible.
Example must be practical, involve choice of route, have arc 'costs'.

**b**

| Stage | State | Action | Value |
|---|---|---|---|
| 1 | *H* | *HK* | 18* |
| | *I* | *IK* | 19* |
| | *J* | *JK* | 21* |
| 2 | *F* | *FH* | Min (16, 18) = 16 |
| | | *FI* | Min (23, 19) = 19* |
| | | *FJ* | Min (17, 21) = 17 |
| | *G* | *GH* | Min (20, 18) = 18 |
| | | *GI* | Min (15, 19) = 15 |
| | | *GJ* | Min (28, 21) = 21* |
| 3 | *B* | *BG* | Min (18, 21) = 18* |
| | *C* | *CF* | Min (25, 19) = 19* |
| | | *CG* | Min (16, 21) = 16 |
| | *D* | *DF* | Min (22, 19) = 19* |
| | | *DG* | Min (19, 21) = 19* |
| | *E* | *EF* | Min (14, 19) = 14* |
| 4 | *A* | *AB* | Min (24, 18) = 18 |
| | | *AC* | Min (25, 19)= 19* |
| | | *AD* | Min (27, 19) = 19* |
| | | *AE* | Min (23, 14) = 14 |

**c** Routes *ACFIK*, *ADFIK*, *ADGJK*

**4 a** *SAEHJT* or *SCEHJT*

**b** 4.7 metres

**c** *SAEHJT*, 5.2 metres

**5**

| **Month** | May | June | July | August | September |
|---|---|---|---|---|---|
| **Production schedule** | 4 | 4 | 5 | 5 | 4 |

Cost £2300

**6 a** £630

**b**

| **Month** | August | September | October | November |
|---|---|---|---|---|
| **Make** | 3 | 4 | 4 | 2 |

Cost = £1540

**c** Profit per cycle = 13 × 1400
= 18 200
Cost of Kris' time = 2000
Cost of production = 1540
⇒ Total profit = 18 200 – 3540
= £14 660

**7 a** Stage – number of weeks to finish
State – show being attended
Action – next journey to undertake

b

| Stage | State | Action | Value |
|---|---|---|---|
| 1 | $F$<br>$G$<br>$H$ | $F$ – Home<br>$G$ – Home<br>$H$ – Home | 500 – 80 = 420*<br>700 – 90 = 610*<br>600 – 70 = 530* |
| 2 | $D$ | $DF$<br>$DG$<br>$DH$ | 1500 – 200 + 420 = 1720<br>1500 – 160 + 610 = 1950*<br>1500 – 120 + 530 = 1910 |
| | $E$ | $EF$<br>$EG$<br>$EH$ | 1300 – 170 + 420 = 1550<br>1300 – 100 + 610 = 1810*<br>1300 – 110 + 530 = 1720 |
| 3 | $A$ | $AD$<br>$AE$ | 900 – 180 + 1950 = 2670*<br>900 – 150 + 1810 = 2560 |
| | $B$ | $BD$<br>$BE$ | 800 – 140 + 1950 = 2610*<br>800 – 120 + 1810 = 2490 |
| | $C$ | $CD$<br>$CE$ | 1000 – 200 + 1950 = 2750*<br>1000 – 210 + 1810 = 2600 |
| 4 | Home | Home – $A$<br>Home – $B$<br>Home – $C$ | –70 + 2670 = 2600*<br>–80 + 2610 = 2530<br>–150 + 2750 = 2600* |

Two optimal schedules:
Home – $ADG$ – Home
Home – $CDG$ – Home

**c** Either shows $ADG$ or shows $CDG$.
In both cases, expected profit is £2600.

**8 a** Each player picks the choice that results in the least worst option.

**b** Play safe is $A$ plays 2 or 4 and $B$ plays 3

**c** The row maximin is –1 and the column minimax is –1. Since row maximin = column minimax, there is a stable solution.
The saddle points are ($A$ plays 2, $B$ plays 3) and ($A$ plays 4, $B$ plays 3)

**d** Value of game to $B$ is –(–1) = 1

**9 a** A game in which the gain to one player is equal to the loss of the other.

**b** Row maximin (–2) ≠ column minimax (1) so there is no stable solution.

**c** Emma should play 1 with probability $\frac{4}{11}$
2 with probability $\frac{7}{11}$
The value of the game is $-\frac{2}{11}$ to Emma.

**d** Value to Freddie is $\frac{2}{11}$

**10 a** 5 is not the smallest value in its row.

**b** The row maximin is 0 and the column minimax is 3. Since row maximin ≠ column minimax, there is no stable solution.

**c i** $B$ should play 1 with probability $\frac{1}{3}$
$B$ should play 1 with probability $\frac{2}{3}$

**ii** The value of the game to player $A$ is $\frac{4}{3}$

**11 a**

| | $A$ plays 4 | $A$ plays 5 |
|---|---|---|
| **$B$ plays 4** | –16 | 20 |
| **$B$ plays 5** | 20 | –25 |

**b** The row maximin is –16 and the column minimax is 20. Since row maximin ≠ column minimax, there is no stable solution.

Amir plays 4 with probability $\frac{5}{9}$ and 5 with probability $\frac{4}{9}$

**c** The value of the game to Amir is 0.

**12 a** Row 1 dominates row 2 so A will never choose row 2. Column 1 dominates column 3 so B will never choose column 3. Thus row 2 and column 3 may be deleted.

**b** $A$ should play row 1 with probability $\frac{3}{5}$
$A$ should play row 2 never
$A$ should play row 3 with probability $\frac{2}{5}$
$B$ should play column 1 with probability $\frac{2}{5}$
$B$ should play column 2 with probability $\frac{3}{5}$
$B$ should play column 3 never.
Value of game is $4\frac{1}{5}$ to $A$

**13 a** Player $A$: play 2
Player $B$: play 1

**b** Row maximin (0) ≠ column minimax (2) so there is no stable solution.

**c** For player $A$ row 2 dominates row 3 (so $A$ will *never* play 3), since $1 > 0$   $3 > 1$   $0 > -3$

**d** $A$ should play 1 with probability $\frac{3}{7}$
2 with probability $\frac{4}{7}$
The value of the game is $\frac{9}{7}$ to $A$

**14 a** In a pure strategy a player always makes the same choice. In a mixed strategy each action is assigned a probability, and is selected with the given probability.

**b**

| | $B$ plays 2 | $B$ plays 3 |
|---|---|---|
| **$A$ plays 1** | 1 | –1 |
| **$A$ plays 2** | –3 | 2 |
| **$A$ plays 3** | –1 | 0 |

**c** The row maximin is –1 and the column minimax is 1. Since row maximin ≠ column minimax, there is no stable solution.

**d** $B$ should play 2 with probability $\frac{5}{7}$ and play 3 with probability $\frac{2}{7}$

**15 a**

| | $A$ plays 1 | $A$ plays 2 |
|---|---|---|
| **$B$ plays 1** | 3 | –4 |
| **$B$ plays 2** | –2 | 1 |
| **$B$ plays 3** | –5 | 4 |

**b** Let $q_1$ be the probability that $B$ plays row 1
Let $q_2$ be the probability that $B$ plays row 2
Let $q_3$ be the probability that $B$ plays row 3
Let value of the game be $v$ and let $V = v + 6$
where $q_1, q_2, q_3 \geqslant 0$
e.g. Maximise $P = V$
subject to $V - 9q - 4q_2 - q_3 + r = 0$
$V - 2q_1 - 7q_2 - 10q_3 + s = 0$
$q_1 + q_2 + q_3 + t = 1$

**16 a** The row maximin is –1 and the column minimax is 1. Since row maximin ≠ column minimax, there is no stable solution.

**b** Row 2 dominates row 3.
Column 1 dominates column 4.
So delete row 3 and column 4.

**c** Let $A$ play row 1 with probability $p_1$, row 2 with probability $p_2$ and row 3 with probability $p_3$

e.g. Maximise $P = V$
subject to
$V - p_1 - 2p_2 - 4p_3 \leqslant 0$
$V - 4p_1 - 6p_2 - p_3 \leqslant 0$
$V - 6p_1 - 5p_2 - 2p_3 \leqslant 0$
$p_1 + p_2 + p_3 \leqslant 1$
$V, p_1, p_2, p_3 \geqslant 0$

**17 a** A zero-sum game is one in which the sum of the gains for all players is zero. (o.e.)

**b** The row maximin is 3 and the column minimax is 4. Since row maximin ≠ column minimax, there is no stable solution.

**c** $A$ should play row 1 $\frac{1}{3}$ of time and row 2 $\frac{2}{3}$ of time; value (to $A$) = $3\frac{2}{3}$

**d** Let $B$ play column 1 with probability $q_1$, column 2 with probability $q_2$ and column 3 with probability $q_3$

Let $v$ = value of game to $B$ so $V = v + 6$

e.g. $\begin{bmatrix} -5 & -3 \\ -2 & -5 \\ -3 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 3 & 2 \end{bmatrix}$

Maximise $P = V$
subject to
$V - q_1 - 4q_2 - 3q_3 \leqslant 0$
$V - 3q_1 - q_2 - 2q_3 \leqslant 0$ $\quad q_1 + q_2 + q_3 \leqslant 1$
$V, q_1, q_2, q_3 \geqslant 0$

**18 a** $u_n = c(4^n) - \frac{1}{3}$ **b** $u_n = \dfrac{4^n(22) - 1}{3}$

**19 a** $p_5 = 35, p_6 = 51$ **b** $p_n = p_{n-1} + 3n - 2$

**c** $p_n = \dfrac{n}{2}(3n - 1)$ **d** 14 950

**20** $u_n = 2^{n+1}(9) - 3n - 7$

**21 a** 103 **b i** $3^n(4) - 5$ **ii** $n = 12$

**22 a** $u_n = (0.8)^4 u_{n-1} + 100$

**b** $u_n = 169.38(1 - (0.8)^{4n})$

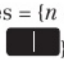
**c** 3

**23 a** $u_n = A(-1)^n + B(5)^n$ **b** $u_n = 10(-1)^n - 2(5)^n$

**24 a** $k = 4$ **b** $u_n = \frac{-57}{14}\left(\frac{2}{3}\right)^n + \frac{1}{14}(-4)^n + 4$

**25 a** 1 way to make a length 1 inch,  so $x_1 = 1$

2 ways to make a length 2 inches,  and  so $x_2 = 2$

$n + 2$ inches = {$n + 1$ inches + } +
{$n$ inches + }
So $x_{n+2} = x_{n+1} + x_n$

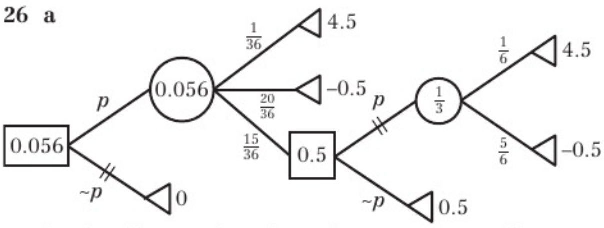
**b** 34

**c i** $u_n = \dfrac{5 + \sqrt{5}}{10}\left(\dfrac{1 + \sqrt{5}}{2}\right)^n + \dfrac{5 - \sqrt{5}}{10}\left(\dfrac{1 - \sqrt{5}}{2}\right)^n$

**ii** 75 025

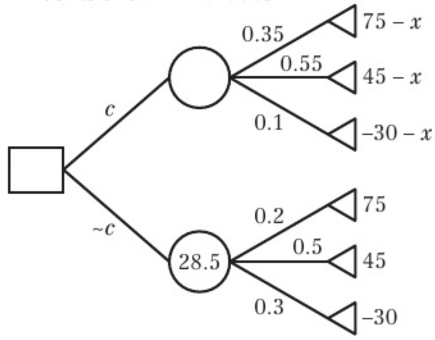
**25** Play the game but if exactly two scores are the same on the first toll of the dice, do not continue. EMV = 5.6 p

**26 a**




**b** Play the game but if exactly two scores are the same on the first roll of the dice, do not continue. E.M.V. = 5.6 p

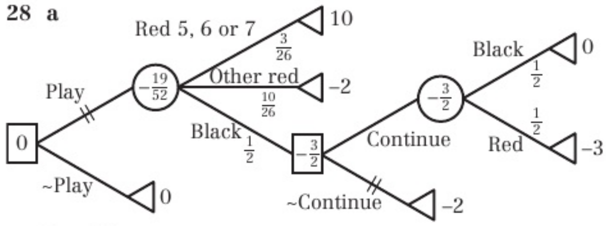
**27 a** Amounts shown in £1000s




**b** $48 - x$, where $x$ is in £1000s

**c** £19 500

**28 a**

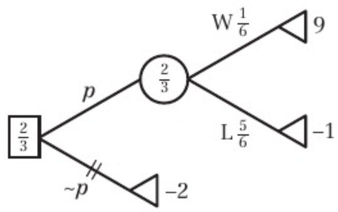



**b** $-\frac{19}{52}$

The best strategy is not to play the game. However, if the game is played and a black card is selected, then the player should continue.

**29** First Saturday: EMV = £4094
Second Saturday: EMV = £4158
Choose the second Saturday.

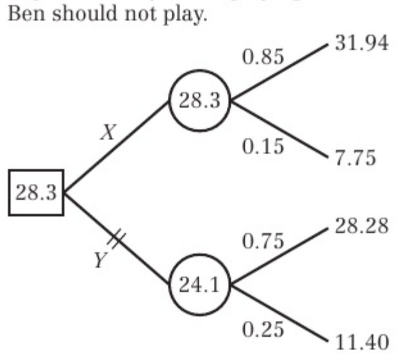
**30 a**




The EMV is $\frac{2}{3}$ so Ben should play the game.

**b** Expected utility of playing = $\sqrt[3]{10^2} \times \frac{1}{6} = 0.774$
Expected utility of not playing = $\sqrt[3]{1} = 1$
Ben should not play.

**31 a**




**b** Project $X$: Expected utility = 28.3
Project $Y$: Expected utility = 24.1
Project $X$ is the better option

**Challenge**

1 **a** 4 choices for each edge, so for $n$ edges there are $4^n$ total choices.

**b** $u_n = 4^{n-1} - u_{n-1}$, $u_1 = 0$. Each closed walk of length $n$ is generated by connecting an open walk of length $u_{n-1}$ back to $A$. There are $4^{n-1}$ total walks of length $n - 1$, so there are $4^{n-1} - u_{n-1}$ open walks of length $n - 1$.

**c** C.F. is $u_n = A(-1)^n$ and P.S. is $u_n = \frac{1}{5}(4^n)$.
Substituting $u_1 = 0$ into $u_n = A(-1)^n + \frac{1}{5}(4^n)$
gives $u_n = \dfrac{4^n + 4(-1)^n}{5}$

**d** Number of closed walks of length $n$ starting and ending at $A$ on $K_p$ satisfies $u_n = (p - 1)^{n-1} - u_{n-1}$, $u_1 = 0$.
Solving gets $u_n = \dfrac{(p-1)^n + (p-1)(-1)^n}{p}$
By symmetry, the number of closed walks starting and ending at *any* vertex is the same, so the total number of closed walks is
$pu_n = (p - 1)^n + (p - 1)(-1)^n$ as required.

2 **a** Play 1 with probability $\frac{4}{15}$, 2 with probability $\frac{1}{15}$ and 3 with probability $\frac{2}{3}$.

**b** Play 1 with probability 0.6, 2 with probability 0.3 and 3 with probability 0.1.

3 12.5% (1 d.p.)

## Exam-style practice: AS level

1 Subtracting each element from 62

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| ***A*** | 11 | 15 | 0 | 12 | 7 |
| ***B*** | 8 | 11 | 2 | 9 | 11 |
| ***C*** | 13 | 10 | 4 | 7 | 9 |
| ***D*** | 10 | 6 | 1 | 4 | 5 |
| ***E*** | 6 | 14 | 3 | 7 | 6 |

Reducing by rows

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| ***A*** | 11 | 15 | 0 | 12 | 7 |
| ***B*** | 6 | 9 | 0 | 7 | 9 |
| ***C*** | 9 | 6 | 0 | 3 | 5 |
| ***D*** | 9 | 5 | 0 | 3 | 4 |
| ***E*** | 3 | 11 | 0 | 4 | 3 |

Reducing by columns

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| ***A*** | 8 | 10 | 0 | 9 | 4 |
| ***B*** | (3) | 4 | 0 | 4 | 6 |
| ~~***C***~~ | ~~6~~ | ~~1~~ | ~~0~~ | ~~0~~ | ~~2~~ |
| ~~***D***~~ | ~~6~~ | ~~0~~ | ~~0~~ | ~~0~~ | ~~1~~ |
| ~~***E***~~ | ~~0~~ | ~~6~~ | ~~0~~ | ~~1~~ | ~~0~~ |

The zeros can be covered with 4 lines.
The smallest uncovered element is 3.
Subtracting 3 from each uncovered element and adding 3 to each element covered twice:

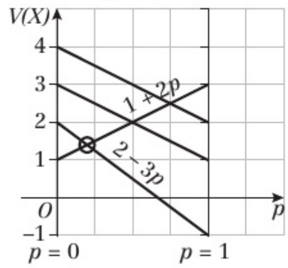
| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| ***A*** | 5 | 7 | 0 | 6 | 1 |
| ***B*** | 0 | 1 | 0 | 1 | 3 |
| ***C*** | 6 | 1 | 3 | 0 | 2 |
| ***D*** | 6 | 0 | 3 | 0 | 1 |
| ***E*** | 0 | 6 | 3 | 1 | 0 |

Five lines are now needed to cover the zeros. So the solution is optimal.
*A*-3, *B*-1, *C*-4, *D*-2, *E*-5
Maximum profit = £283

2 **a** A game in which each player's gain or loss is balanced by the losses or gains of the other players.

**b** Column minimax = 2 ≠ row maximin = 1, so no stable solution.

**c** Assume player $X$ plays 1 with probability $p$ and 2 with probability $(1 - p)$.
If $Y$ plays 1, value of game to $X$ is
$3p + 1 - p = 1 + 2p$
If $Y$ plays 2, value of game to $X$ is
$-p + 2(1 - p) = 2 - 3p$
If $Y$ plays 3, value of game to $X$ is
$2p + 4(1 - p) = 4 - 2p$
If $Y$ plays 4, value of game to $X$ is
$p + 3(1 - p) = 3 - 2p$




Optimum strategy for $X$ occurs when $1 + 2p = 2 - 3p$
$\Rightarrow p = \frac{1}{5}$
Value of the game to $X$ is $1\frac{2}{5}$

3 **a** Because it does not separate the source from the sink.

**b** **i** 87 **ii** $x = 14$ **iii** 80

**c** *SACDGT*

**d** Flow is now 83 = capacity of cut through *CF*, *DF*, *DG* and *EG*.
By the maximum flow–minimum cut theorem, the flow is now maximal.

4 **a** $L_n = \left(1 + \dfrac{r}{100}\right)L_{n-1} - p$

**b** Let $\lambda$ be a particular solution
then $\lambda = \left(1 + \dfrac{r}{100}\right)\lambda - p$
$\lambda = \dfrac{100p}{r}$

**c** General solution is $L_n = k\left(1 + \dfrac{r}{100}\right)^n + \dfrac{100p}{r}$
Since $L_1 = X\left(1 + \dfrac{r}{100}\right) - \dfrac{100p}{r}$,
Since $L_0 = X$, $\quad k + \dfrac{100p}{r} = X$
$\Rightarrow k = X - \dfrac{100p}{r}$
$L_n = \left(X - \dfrac{100p}{r}\right)\left(1 + \dfrac{r}{100}\right)^n + \dfrac{100p}{r}$



Online Full worked solutions are available in SolutionBank.

**d** After $n$ payments $L_n = 0$

$$\left(X - \frac{100p}{r}\right)\left(1 + \frac{r}{100}\right)^n + \frac{100p}{r} = 0$$

$$\left(\frac{Xr - 100p}{r}\right)\left(\frac{r+100}{100}\right)^n + \frac{100p}{r} = 0$$

$$(Xr - 100p)\left(\frac{r+100}{100}\right)^n + 100p = 0$$

$$(Xr - 100p)(r+100)^n + 100^{n+1}p = 0$$

$$Xr(r+100)^n = p(100(r+100)^n - 100^{n+1})$$

$$p = \frac{Xr(r+100)^n}{100(r+100)^n - 100^{n+1}} = \frac{Xr}{100 - 100^{n+1}(r+100)^{-n}}$$

## Exam-style practice: A level

**1 a** The auxiliary equation is $m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$ so the equation has a repeated root of 3.

The general solution of the recurrence relation is $u_n = A(3)^n + Bn(3)^n$

**b** Let $\lambda$ be a particular solution of the recurrence relation. Then

$\lambda - 6\lambda + 9\lambda = 15 \Rightarrow \lambda = \frac{15}{4}$

The general solution is $u_n = A(3)^n + Bn(3)^n + \frac{15}{4}$

$u_1 = 3A + 3B + \frac{15}{4} = \frac{21}{4} \Rightarrow A + B = \frac{1}{2}$ (1)

$u_2 = 9A + 18B + \frac{15}{4} = -\frac{93}{4} \Rightarrow A + 2B = -3$ (2)

(2) – (1) gives $B = -\frac{7}{2}$, $A = 4$

So $u_n = 4(3)^n - \frac{7n}{2}(3)^n + \frac{15}{4}$

**2 a** Every number in column 3 is smaller than the corresponding number in column 2. Thus $Y$ would never play 2 and the column may be removed from the pay-off matrix. This gives the reduced matrix:

| | ***Y* plays 1** | ***Y* plays 3** |
|---|---|---|
| ***X* plays 1** | 3 | 1 |
| ***X* plays 2** | 1 | 2 |
| ***X* plays 3** | –2 | 1 |
| ***X* plays 4** | 1 | 4 |

**b**

| | ***Y* plays 1** | ***Y* plays 3** | **Row minimum** |
|---|---|---|---|
| ***X* plays 1** | 3 | 1 | 1 |
| ***X* plays 2** | 1 | 2 | 1 |
| ***X* plays 3** | –2 | 1 | –2 |
| ***X* plays 4** | 1 | 4 | 1 |
| **Column maximum** | 3 | 4 | |

Row maximin = 1, Column minimax = 3

Row maximin ≠ column minimax so there is no stable solution and mixed strategies should be used.

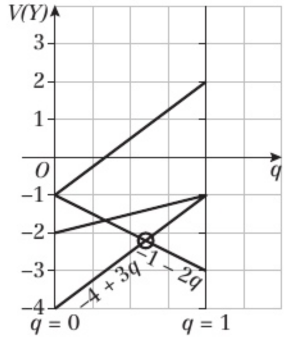
**c** If $Y$ plays 1 with probability $q$ and plays 3 with probability $1 - q$, then:

If $X$ plays 1, the pay-off for $Y$ is $-(3q + (1-q)) = -1 - 2q$

If $X$ plays 2, the pay-off for $Y$ is $-(q + 2(1-q)) = -2 + q$

If $X$ plays 3, the pay-off for $Y$ is $-(2q + (1-q)) = -1 + 3q$

If $X$ plays 4, the pay-off for $Y$ is $-(q + 4(1-q)) = -4 + 3q$




$Y$ should choose $q$ such that $-4 + 3q = -1 - 2q$

$\Rightarrow q = \frac{3}{5}$

Value of the game to $B$ is $-4 + 3 \times \frac{3}{5} = -\frac{11}{5}$

**3 a** Reducing rows:

| | | | |
|---|---|---|---|
| 0 | 6 | 17 | 12 |
| 0 | 2 | 5 | 3 |
| 0 | 11 | 8 | 13 |
| 0 | 8 | 2 | 6 |

Reducing columns:

| | | | |
|---|---|---|---|
| 0 | ④ | 15 | 9 |
| 0 | 0 | 3 | 0 |
| 0 | 9 | 6 | 10 |
| 0 | 6 | 0 | 3 |

The zeros can be covered with 3 lines, so the solution is not optimal. The smallest uncovered element is 4.

Augmenting by 4:

| | | | |
|---|---|---|---|
| 0 | 0 | 11 | 5 |
| 4 | 0 | 3 | 0 |
| 0 | 5 | 2 | 6 |
| 4 | 6 | 0 | 3 |

Four lines are now required to cover the zeros, so the solution is optimal.

$P \to X$, $Q \to Z$, $R \to W$, $S \to Y$

Total cost = £168

**b** $x_{ij} = \begin{cases} 1 & \text{if worker } i \text{ does task } j \\ 0 & \text{otherwise} \end{cases}$

where $i \in \{P, Q, R, S\}$ and $j \in \{W, X, Y, Z\}$ and $x_{ij} \geqslant 0$

Minimise

$35x_{PW} + 41x_{PX} + 52x_{PY} + 47x_{PZ}$

$+ 43x_{QW} + 45x_{QX} + 48x_{QY} + 46x_{QZ}$

$+ 37x_{RW} + 48x_{RX} + 45x_{RY} + 50x_{RZ}$

$+ 42x_{SW} + 50x_{SX} + 44x_{SY} + 48x_{SZ}$

subject to $\sum x_{iW} = 1$, $\sum x_{iX} = 1$, $\sum x_{iY} = 1$, $\sum x_{iZ} = 1$,

$\sum x_{Pj} = 1$, $\sum x_{Qj} = 1$, $\sum x_{Rj} = 1$, $\sum x_{Sj} = 1$

**4 a** Using the north-west corner method gives:

| | *P* | *Q* | *R* | |
|---|---|---|---|---|
| *A* | 10 | – | – | 10 |
| *B* | 2 | 14 | 0 | 16 |
| *C* | – | – | 14 | 14 |
| | 12 | 14 | 14 | |

To avoid a degenerate solution, a zero could either be placed in cell *BR* (as above) or in cell *CQ*.
Total cost = £460

**b** Shadow costs and improvement indices:

| | | 11 | 7 | 11 |
|---|---|---|---|---|
| | | *P* | *Q* | *R* |
| 0 | *A* | × | 5 | 4 |
| 3 | *B* | × | × | × |
| 2 | *C* | –1 | 7 | × |

There is only one negative improvement index, which is in cell *CP*. *CP* is the entering cell.

| | *P* | *Q* | *R* |
|---|---|---|---|
| *A* | 10 | | |
| *B* | $2-\theta$ | 14 | $\theta$ |
| *C* | $\theta$ | | $14-\theta$ |

Taking $\theta = 2$, *BP* is the exiting cell.
The improved solution is

| | *P* | *Q* | *R* |
|---|---|---|---|
| *A* | 10 | – | – |
| *B* | – | 14 | 2 |
| *C* | 2 | – | 12 |

Shadow costs and improvement indices:

| | | 11 | 8 | 12 |
|---|---|---|---|---|
| | | *P* | *Q* | *R* |
| 0 | *A* | × | 4 | 3 |
| 2 | *B* | 1 | × | × |
| 1 | *C* | × | 7 | × |

There are no negative improvement indices so the solution is optimal.
Total cost = £458

**5 a** $w = 24$, $x = 10$, $y = 17$ and $z = 33$

**b i** $25 + 10 + 14 + 35 = 84$

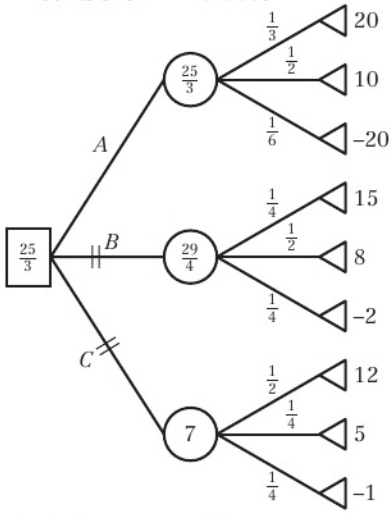
**ii** $35 - 4 + 14 + 35 = 80$

**c** The maximum flow is less than or equal to 80.

**d** *SBADT* (+2) and *SCFT* (+2)

**e** Value of cut through *SA*, *SB* and *SC* has value $24 + 33 + 20 = 77$. The initial flow had value 73. The augmented flow has value $73 + 4 = 77$. By the maximum flow – minimum cut theorem, the flow of 77 is maximal.

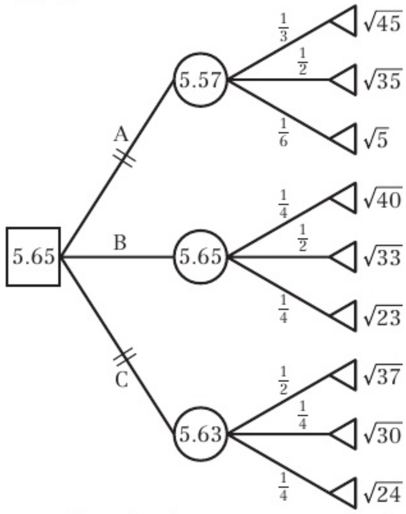
**6 a** Amounts shown in £1000s




The highest EMV is $\frac{25}{3}$ suggesting that Project *A* is the best option.

**b** A small start-up company may not have the resources to withstand the possible loss associated with project *A*. A utility function could be chosen to reflect a degree of risk aversion, making it more appropriate than the EMVs alone.

**c**




**d** Using the utility function, Project *B* is the favoured option.

7

| Stage | State | Action | Destination | Value |
|---|---|---|---|---|
| October (15) | 2 | 13 | 0 | £300+ £10 000 = £10 300* |
| | 1 | 14 | 0 | £150+ £10 000 = £10 150* |
| | 0 | 15 | 0 | £800+ £10 000 = £10 800* |
| September (17) | 2 | 15 | 0 | £300 + £800 + £10 000 + £10 800 = £21 900 |
| | | 16 | 1 | £300 + £800 + £10 000 + £10 150 = £21 250* |
| | | 17 | 2 | £300 + £800 + £10 000 + £10 300 = £21 400 |
| | 1 | 16 | 0 | £150 + £800 + £10 000 + £10 800 = £21 750 |
| | | 17 | 1 | £150 + £800 + £10 000 + £10 150 = £21 100* |
| | 0 | 17 | 0 | £800 + £10 000 + £10 800 = £21 600* |
| August (13) | 2 | 11 | 0 | £300 + £10 000 + £21 600 = £31 900 |
| | | 12 | 1 | £300 + £10 000 + £21 100 = £31 400* |
| | | 13 | 2 | £300 + £10 000 + £21 250 = £31 550 |
| | 1 | 12 | 0 | £150 + £10 000 + £21 600 = £31 750 |
| | | 13 | 1 | £150 + £10 000 + £21 100 = £31 250* |
| | | 14 | 2 | £150 + £10 000 + £21 250 = £31 400 |
| | 0 | 13 | 0 | £10 000 + £21 600 = £31 600 |
| | | 14 | 1 | £10 000 + £21 100 = £31 100* |
| | | 15 | 2 | £800 + £10 000 + £21 250 = £32 050 |
| July (15) | 0 | 15 | 0 | £800 + £10 000 + £31 100 = £41 900* |
| | | 16 | 1 | £800 + £10 000 + £31 250 = £42 050 |
| | | 17 | 2 | £800 + £10 000 + £31 400 = £42 200 |

| **Month** | July | August | September | October |
|---|---|---|---|---|
| **Number made** | 15 | 14 | 17 | 14 |

Minimum production cost = £41 900

# Index