Algebraic methods 7B

1 a
$$x+1\overline{)\,x^3+6x^2+8x+3}$$
$$\begin{array}{r} x^2+5x+3 \\ \hline x^3+x^2 \\ \underline{\quad\quad} \\ 5x^2+8x \\ \underline{5x^2+5x} \\ 3x+3 \\ \underline{3x+3} \\ 0 \end{array}$$
So $\dfrac{x^3+6x^2+8x+3}{x+1} = (x+1)(x^2+5x+3)$

b
$$x+4\overline{)\,x^3+10x^2+25x+4}$$
$$\begin{array}{r} x^2+6x+1 \\ \hline x^3+4x^2 \\ \underline{\quad\quad} \\ 6x^2+25x \\ \underline{6x^2+24x} \\ x+4 \\ \underline{x+4} \\ 0 \end{array}$$
So $\dfrac{x^3+10x^2+25x+4}{x+4} = (x+4)(x^2+6x+1)$

c
$$x+2\overline{)\,x^3-x^2+x+14}$$
$$\begin{array}{r} x^2-3x+7 \\ \hline x^3+2x^2 \\ \underline{\quad\quad} \\ -3x^2+x \\ \underline{-3x^2-6x} \\ 7x+14 \\ \underline{7x+14} \\ 0 \end{array}$$
So $\dfrac{x^3-x^2+x+14}{x+2} = (x+2)(x^2-3x+7)$

d
$$x-3\overline{)\,x^3+x^2-7x-15}$$
$$\begin{array}{r} x^2+4x+5 \\ \hline x^3-3x^2 \\ \underline{\quad\quad} \\ 4x^2-7x \\ \underline{4x^2-12x} \\ 5x-15 \\ \underline{5x-15} \\ 0 \end{array}$$
So $\dfrac{x^3+x^2-7x-15}{x-3} = (x-3)(x^2+4x+5)$

e
$$x-5\overline{)\,x^3-8x^2+13x+10}$$
$$\begin{array}{r} x^2-3x-2 \\ \hline x^3-5x^2 \\ \underline{\quad\quad} \\ -3x^2+13x \\ \underline{-3x^2+15x} \\ -2x+10 \\ \underline{-2x+10} \\ 0 \end{array}$$
So $\dfrac{x^3-8x^2+13x+10}{x-5} = (x-5)(x^2-3x-2)$

f
$$x-7\overline{)\,x^3-5x^2-6x-56}$$
$$\begin{array}{r} x^2+2x+8 \\ \hline x^3-7x^2 \\ \underline{\quad\quad} \\ 2x^2-6x \\ \underline{2x^2-14x} \\ 8x-56 \\ \underline{8x-56} \\ 0 \end{array}$$
So $\dfrac{x^3-5x^2-6x-56}{x-7} = (x-7)(x^2+2x+8)$

2 a
$$\begin{array}{r} 6x^2+3x+2 \\ x+4\overline{)\ 6x^3+27x^2+14x+8} \\ \underline{6x^3+24x^2} \\ 3x^2+14x \\ \underline{3x^2+12x} \\ 2x+8 \\ \underline{2x+8} \\ 0 \end{array}$$

So $6x^3+27x^2+14x+8 = (x+4)(6x^2+3x+2)$

b
$$\begin{array}{r} 4x^2+x-5 \\ x+2\overline{)\ 4x^3+9x^2-3x-10} \\ \underline{4x^3+8x^2} \\ x^2-3x \\ \underline{x^2+2x} \\ -5x-10 \\ \underline{-5x-10} \\ 0 \end{array}$$

So $4x^3+9x^2-3x-10 = (x+2)(4x^2+x-5)$

c
$$\begin{array}{r} 2x^2-2x-3 \\ x+3\overline{)\ 2x^3+4x^2-9x-9} \\ \underline{2x^3+6x^2} \\ -2x^2-9x \\ \underline{-2x^2-6x} \\ -3x-9 \\ \underline{-3x-9} \\ 0 \end{array}$$

So $2x^3+4x^2-9x-9 = (x+3)(2x^2-2x-3)$

d
$$\begin{array}{r} 2x^2-3x-4 \\ x-6\overline{)\ 2x^3-15x^2+14x+24} \\ \underline{2x^3-12x^2} \\ -3x^2+14x \\ \underline{-3x^2+18x} \\ -4x+24 \\ \underline{-4x+24} \\ 0 \end{array}$$

So $2x^3-15x^2+14x+24 = (x-6)(2x^2-3x-4)$

e
$$\begin{array}{r} -5x^2+3x+5 \\ x+6\overline{)\ -5x^3-27x^2+23x+30} \\ \underline{-5x^3-30x^2} \\ 3x^2+23x \\ \underline{3x^2+18x} \\ 5x+30 \\ \underline{5x+30} \\ 0 \end{array}$$

So $-5x^3-27x^2+23x+30 = (x+6)(-5x^2+3x+5)$

f
$$\begin{array}{r} -4x^2+x-1 \\ x-2\overline{)\ -4x^3+9x^2-3x+2} \\ \underline{-4x^3+8x^2} \\ x^2-3x \\ \underline{x^2-2x} \\ -x+2 \\ \underline{-x+2} \\ 0 \end{array}$$

So $-4x^3+9x^2-3x+2 = (x-2)(-4x^2+x-1)$

3 a
$$x+2\overline{)\,x^4+5x^3+2x^2-7x+2}$$
$$\begin{array}{r} x^3+3x^2-4x+1 \\ \hline x^4+2x^3 \\ 3x^3+2x^2 \\ \underline{3x^3+6x^2} \\ -4x^2-7x \\ \underline{-4x^2-8x} \\ x+2 \\ \underline{x+2} \\ 0 \end{array}$$
So $\dfrac{x^4+5x^3+2x^2-7x+2}{x+2}$

$= x^3+3x^2-4x+1$

b
$$x+3\overline{)\,4x^4+14x^3+3x^2-14x-15}$$
$$\begin{array}{r} 4x^3+2x^2-3x-5 \\ \hline 4x^4+12x^3 \\ 2x^3+3x^2 \\ \underline{2x^3+6x^2} \\ -3x^2-14x \\ \underline{-3x^2-9x} \\ -5x-15 \\ \underline{-5x-15} \\ 0 \end{array}$$
So $\dfrac{4x^4+14x^3+3x^2-14x-15}{x+3}$

$= 4x^3+2x^2-3x-5$

c
$$x-2\overline{)\,-3x^4+9x^3-10x^2+x+14}$$
$$\begin{array}{r} -3x^3+3x^2-4x-7 \\ \hline -3x^4+6x^3 \\ 3x^3-10x^2 \\ \underline{3x^3-6x^2} \\ -4x^2+x \\ \underline{-4x^2+8x} \\ -7x+14 \\ \underline{-7x+14} \\ 0 \end{array}$$
c So $\dfrac{-3x^4+9x^3-10x^2+x+14}{x-2}$

$= -3x^3+3x^2-4x-7$

d
$$x-1\overline{)\,-5x^5+7x^4+2x^3-7x^2+10x-7}$$
$$\begin{array}{r} -5x^4+2x^3+4x^2-3x+7 \\ \hline -5x^5+5x^4 \\ 2x^4+2x^3 \\ \underline{2x^4-2x^3} \\ 4x^3-7x^2 \\ \underline{4x^3-4x^2} \\ -3x^2+10x \\ \underline{-3x^2+3x} \\ 7x-7 \\ \underline{7x-7} \\ 0 \end{array}$$
So $\dfrac{-5x^5+7x^4+2x^3-7x^2+10x-7}{x-1}$

$= -5x^4+2x^3+4x^2-3x+7$

4 a
$$3x+2\overline{)\,3x^4+8x^3-11x^2+2x+8}$$
$$\begin{array}{r} x^3+2x^2-5x+4 \\ \hline 3x^4+2x^3 \\ 6x^3-11x^2 \\ \underline{6x^3+4x^2} \\ -15x^2+2x \\ \underline{-15x^2-10x} \\ 12x+8 \\ \underline{12x+8} \\ 0 \end{array}$$
So $\dfrac{3x^4+8x^3-11x^2+2x+8}{3x+2}$

$= x^3+2x^2-5x+4$

4 b
$$4x+1\overline{)\,4x^4-3x^3+11x^2-x-1}\quad x^3-x^2+3x-1$$
$$\begin{array}{r} 4x^4+x^3 \\ -4x^3+11x^2 \\ \underline{-4x^3-x^2} \\ 12x^2-x \\ \underline{12x^2+3x} \\ -4x-1 \\ \underline{-4x-1} \\ 0 \end{array}$$
So $\dfrac{4x^4-3x^3+11x^2-x-1}{4x+1}$

$= x^3 - x^2 + 3x - 1$

c
$$2x-3\overline{)\,4x^4-6x^3+10x^2-11x-6}\quad 2x^3+5x+2$$
$$\begin{array}{r} 4x^4-6x^3 \\ 0+10x^2-11x \\ \underline{10x^2-15x} \\ 4x-6 \\ \underline{4x-6} \\ 0 \end{array}$$
So $\dfrac{4x^4-6x^3+10x^2-11x-6}{2x-3}$

$= 2x^3 + 5x + 2$

d
$$2x+3\overline{)\,6x^5+13x^4-4x^3-9x^2+21x+18}\quad 3x^4+2x^3-5x^2+3x+6$$
$$\begin{array}{r} 6x^5+9x^4 \\ 4x^4-4x^3 \\ \underline{4x^4+6x^3} \\ -10x^3-9x^2 \\ \underline{10x^3-15x^2} \\ 6x^2+21x \\ \underline{6x^2+9x} \\ 12x+18 \\ \underline{12x+18} \\ 0 \end{array}$$

d So $\dfrac{6x^5+13x^4-4x^3-9x^2+21x+18}{2x+3}$

$= 3x^4 + 2x^3 - 5x^2 + 3x + 6$

e
$$3x-1\overline{)\,6x^5-8x^4+11x^3+9x^2-25x+7}\quad 2x^4-2x^3+3x^2+4x-7$$
$$\begin{array}{r} 6x^5-2x^4 \\ -6x^4+11x^3 \\ \underline{-6x^4+2x^3} \\ 9x^3+9x^2 \\ \underline{9x^3-3x^2} \\ 12x^2-25x \\ \underline{12x^2-4x} \\ -21x+7 \\ \underline{-21x+7} \\ 0 \end{array}$$
So $\dfrac{6x^5-8x^4+11x^3+9x^2-25x+7}{3x-1}$

$= 2x^4 - 2x^3 + 3x^2 + 4x - 7$

f
$$2x-5\overline{)\,8x^5-26x^4+11x^3+22x^2-40x+25}\quad 4x^4-3x^3-2x^2+6x-5$$
$$\begin{array}{r} 8x^5-20x^4 \\ -6x^4+11x^3 \\ \underline{-6x^4+15x^3} \\ -4x^3+22x^2 \\ \underline{-4x^3+10x^2} \\ 12x^2-40x \\ \underline{12x^2-30x} \\ -10x+25 \\ \underline{-10x+25} \\ 0 \end{array}$$
So $\dfrac{8x^5-26x^4+11x^3+22x^2-40x+25}{2x-5}$

$= 4x^4 - 3x^3 - 2x^2 + 6x - 5$

4 g

$$5x+3\overline{)\,25x^4+75x^3+6x^2-28x-6}$$

$$\begin{array}{r} 5x^3+12x^2-6x-2 \\ \hline 25x^4+15x^3 \\ 60x^3+6x^2 \\ \underline{60x^3+36x^2} \\ -30x^2-28x \\ \underline{-30x^2-18x} \\ -10x-6 \\ \underline{-10x-6} \\ 0 \end{array}$$

So $\dfrac{25x^4+75x^3+6x^2-28x-6}{5x+3}$

$= 5x^3 + 12x^2 - 6x - 2$

h

$$7x-2\overline{)\,21x^5+29x^4-10x^3+42x-12}$$

$$\begin{array}{r} 3x^4+5x^3+6 \\ \hline 21x^5-6x^4 \\ 35x^4-10x^3 \\ \underline{35x^4-10x^3} \\ 0+42x-12 \\ \underline{42x-12} \\ 0 \end{array}$$

So $\dfrac{21x^5+29x^4-10x^3+42x-12}{7x-2}$

$= 3x^4 + 5x^3 + 6$

5 a

$$x+2\overline{)\,x^3+0x^2+x+10}$$

$$\begin{array}{r} x^2-2x+5 \\ \hline x^3+2x^2 \\ -2x^2+x \\ \underline{-2x^2-4x} \\ 5x+10 \\ \underline{5x+10} \\ 0 \end{array}$$

So $\dfrac{x^3+x+10}{x+2} = x^2-2x+5$

5 b

$$x+3\overline{)\,2x^3+0x^2-17x+3}$$

$$\begin{array}{r} 2x^2-6x+1 \\ \hline 2x^3+6x^2 \\ -6x^2-17x \\ \underline{-6x^2-18x} \\ x+3 \\ \underline{x+3} \\ 0 \end{array}$$

So $\dfrac{2x^3-17x+3}{x+3} = 2x^2-6x+1$

c

$$x-4\overline{)\,-3x^3+0x^2+50x-8}$$

$$\begin{array}{r} -3x^2-12x+2 \\ \hline -3x^3+12x^2 \\ -12x^2+50x \\ \underline{-12x^2+48x} \\ 2x-8 \\ \underline{2x-8} \\ 0 \end{array}$$

So $\dfrac{-3x^3+50x-8}{x-4} = -3x^2-12x+2$

6 a

$$x-3\overline{)\,x^3+x^2+0x-36}$$

$$\begin{array}{r} x^2+4x+12 \\ \hline x^3-3x^2 \\ 4x^2+0x \\ \underline{4x^2-12x} \\ 12x-36 \\ \underline{12x-36} \\ 0 \end{array}$$

So $\dfrac{x^3+x^2-36}{x-3} = x^2+4x+12$

6 b
$$\begin{array}{r} 2x^2 - x + 5 \\ x+5 \overline{)\ 2x^3 + 9x^2 + 0x + 25} \\ \underline{2x^3 + 10x^2} \\ -x^2 + 0x \\ \underline{-x^2 - 5x} \\ 5x + 25 \\ \underline{5x + 25} \\ 0 \end{array}$$

So $\dfrac{2x^3 + 9x^2 + 25}{x+5} = 2x^2 - x + 5$

c
$$\begin{array}{r} -3x^2 + 5x + 10 \\ x-2 \overline{)\ -3x^3 + 11x^2 + 0x - 20} \\ \underline{-3x^3 + 6x^2} \\ 5x^2 + 0x \\ \underline{5x^2 - 10x} \\ 10x - 20 \\ \underline{10x - 20} \\ 0 \end{array}$$

So $\dfrac{-3x^3 + 11x^2 - 20}{x-2} = -3x^2 + 5x + 10$

7 RHS = $(x + 2)(x^2 - 5)$
$= x^3 + 2x^2 - 5x - 10$
= LHS

8 a
$$\begin{array}{r} x^2 - x + 2 \\ x+5 \overline{)\ x^3 + 4x^2 - 3x + 2} \\ \underline{x^3 + 5x^2} \\ -x^2 - 3x \\ \underline{-x^2 - 5x} \\ 2x + 2 \\ \underline{2x + 10} \\ -8 \end{array}$$

So the remainder is −8.

8 b
$$\begin{array}{r} 3x^2 - 2x - 2 \\ x-6 \overline{)\ 3x^3 - 20x^2 + 10x + 5} \\ \underline{3x^3 - 18x^2} \\ -2x^2 + 10x \\ \underline{-2x^2 + 12x} \\ -2x + 5 \\ \underline{-2x + 12} \\ -7 \end{array}$$

So the remainder is −7.

c
$$\begin{array}{r} -2x^2 - 5x - 8 \\ x-4 \overline{)\ -2x^3 + 3x^2 + 12x + 20} \\ \underline{-2x^3 + 8x^2} \\ -5x^2 + 12x \\ \underline{-5x^2 + 20x} \\ -8x + 20 \\ \underline{-8x + 32} \\ -12 \end{array}$$

So the remainder is −12.

9
$$\begin{array}{r} 3x^2 + x + 1 \\ x-1 \overline{)\ 3x^3 - 2x^2 + 0x + 4} \\ \underline{3x^3 - 3x^2} \\ x^2 + 0x \\ \underline{x^2 - x} \\ x + 4 \\ \underline{x - 1} \\ 5 \end{array}$$

So the remainder is 5.

10
$$\begin{array}{r} 3x^3 - 11x^2 + 21x - 24 \\ x+1 \overline{)\ 3x^4 - 8x^3 + 10x^2 - 3x - 25} \\ \underline{3x^4 + 3x^3} \\ -11x^3 + 10x^2 \\ \underline{-11x^3 - 11x^2} \\ 21x^2 - 3x \\ \underline{21x^2 + 21x} \\ -24x - 25 \\ \underline{-24x - 24} \\ -1 \end{array}$$

10 So the remainder is −1.

11
$$x+4\overline{)\,5x^3+0x^2-73x+28}\qquad 5x^2-20x+7$$
$$\underline{5x^3+20x^2}$$
$$-20x^2-73x$$
$$\underline{-20x^2-80x}$$
$$7x+28$$
$$\underline{7x+28}$$
$$0$$

The remainder is 0, so $(x + 4)$ is a factor of $5x^3 - 73x + 28$.

12
$$x-2\overline{)\,3x^3+0x^2-8x-8}\qquad 3x^2+6x+4$$
$$\underline{3x^3-6x^2}$$
$$6x^2-8x$$
$$\underline{6x^2-12x}$$
$$4x-8$$
$$\underline{4x-8}$$
$$0$$

So $\dfrac{3x^3-8x-8}{x-2} = 3x^2+6x+4$

13
$$x-1\overline{)\,x^3+0x^2+0x-1}\qquad x^2+x+1$$
$$\underline{x^3-x^2}$$
$$x^2+0x$$
$$\underline{x^2-x}$$
$$x-1$$
$$\underline{x-1}$$
$$0$$

So $\dfrac{x^3-1}{x-1} = x^2+x+1$

14
$$x+2\overline{)\,x^4+0x^3+0x^2+0x-16}\qquad x^3-2x^2+4x-8$$
$$\underline{x^4+2x^3}$$
$$-2x^3+0x^2$$
$$\underline{-2x^3-4x^2}$$
$$4x^2+0x$$
$$\underline{4x^2+8x}$$
$$-8x-16$$
$$\underline{-8x-16}$$
$$0$$

So $\dfrac{x^4-16}{x+2} = x^3-2x^2+4x-8$

15
$$5x+4\overline{)\,10x^3+43x^2-2x-10}\qquad 2x^2+7x-6$$
$$\underline{10x^3+8x^2}$$
$$35x^2-2x$$
$$\underline{35x^2+28x}$$
$$-30x-10$$
$$\underline{-30x-24}$$
$$14$$

So the remainder is 14.

16 a
$$x-3\overline{)\,3x^3-14x^2-47x-14}\qquad 3x^2-5x-62$$
$$\underline{3x^3-9x^2}$$
$$-5x^2-47x$$
$$\underline{-5x^2+15x}$$
$$-62x-14$$
$$\underline{-62x+186}$$
$$-200$$

So the remainder is −200.

16 b
$$x+2\overline{)3x^3-14x^2-47x-14}$$
$$\begin{array}{r} 3x^2-20x-7 \\ \hline 3x^3+6x^2 \\ -20x^2-47x \\ \underline{-20x^2-40x} \\ -7x-14 \\ \underline{-7x-14} \\ 0 \end{array}$$

$f(x) = 3x^3 - 14x^2 - 47x - 14$
$= (x + 2)(3x^2 - 20x - 7)$
$= (x + 2)(3x + 1)(x - 7)$

17 a i
$$x-2\overline{)x^3+6x^2+5x-12}$$
$$\begin{array}{r} x^2+8x+21 \\ \hline x^3-2x^2 \\ 8x^2+5x \\ \underline{8x^2-16x} \\ 21x-12 \\ \underline{21x-42} \\ 30 \end{array}$$

So the remainder is 30.

ii
$$x+3\overline{)x^3+6x^2+5x-12}$$
$$\begin{array}{r} x^2+3x-4 \\ \hline x^3+3x^2 \\ 3x^2+5x \\ \underline{3x^2+9x} \\ -4x-12 \\ \underline{-4x-12} \\ 0 \end{array}$$

So the remainder is 0.

b $f(x) = x^3 + 6x^2 + 5x - 12$
$= (x + 3)(x^2 + 3x - 4)$
$= (x + 3)(x + 4)(x - 1)$
So $x = -3$, $x = -4$ or $x = 1$

18 a
$$2x-1\overline{)2x^3+3x^2-8x+3}$$
$$\begin{array}{r} x^2+2x-3 \\ \hline 2x^3-x^2 \\ 4x^2-8x \\ \underline{4x^2-2x} \\ -6x+3 \\ \underline{-6x+3} \\ 0 \end{array}$$

$f(x) = 2x^3 + 3x^2 - 8x + 3$
$= (2x - 1)(x^2 + 2x - 3)$
$a = 1$, $b = 2$, $c = -3$

b $f(x) = 2x^3 + 3x^2 - 8x + 3$
$= (2x - 1)(x^2 + 2x - 3)$
$= (2x - 1)(x - 1)(x + 3)$

c $(2x - 1)(x - 1)(x + 3) = 0$
$x = \frac{1}{2}$, $x = 1$ and $x = -3$

19 a
$$4x-1\overline{)12x^3+5x^2+2x-1}$$
$$\begin{array}{r} 3x^2+2x+1 \\ \hline 12x^3-3x^2 \\ 8x^2+2x \\ \underline{8x^2-2x} \\ 4x-1 \\ \underline{4x-1} \\ 0 \end{array}$$

$f(x) = (4x - 1)(3x^2 + 2x + 1)$
$a = 3$, $b = 2$, $c = 1$

b $(4x - 1)(3x^2 + 2x + 1) = 0$
Using the discriminant for $3x^2 + 2x + 1$:
$b^2 - 4ac = 2^2 - 4(3)(1) = -8 < 0$ so there are no real solutions.
So $f(x)$ has only one real solution, $x = \frac{1}{4}$.